RELATIONS & FUNCTIONS

In this unit you will learn definitions of **relations**, **functions**, **domain** and **range**. You will also learn to perform basic operations on functions and examine the ideas of the '**zeros**' of a function.

Basic Definitions of Relation, Function, Domain and Range

The Algebra of Functions and Composition of Functions

Basic Definitions & Notation

A “function” is a way to describe relationships between two or more mathematical entities or sets. Functions are used to present information about those mathematical relationships. In this unit we will examine relations and functions between two real sets.

Relation

Given two sets, A and $B \in \mathbb{R}$, a ‘relation’ R exists between A and B if R is the set of ordered pairs:

$R = \{(a_1,b_1);(a_2,b_2);(a_3,b_3);\ldots;(a_k,b_k);\ldots\}$ and every $a_i \in A;\ b_i \in B$ and for all $a_n \le a_m;\ b_n \le b_m$ and $\varnothing \notin (a_n,b_n)$

Example #1: $A = \{4,5,11,21\};\ B = \{2,3,8\}$, then $R = \{(4,2);(4,8);(11,2);(21,3)\}$ is a relation between A and B.

Note: A relation between two sets does not have to contain all elements of either set, and the relation may repeat elements from either set.

Example #2: $P = \{5,6,7\};\ Q = \{\ \}$, then $R = \{(5,\varnothing);(6,\varnothing);(7,\varnothing)\}$ is not a relation, since for any element of R there is no element from Q.

Domain and Range

Given a relation $R \in \mathbb{R}$ of the form

$R = \{(a_1,b_1);(a_2,b_2);(a_3,b_3);\ldots;(a_k,b_k);\ldots\}$, then the Domain of R (denoted R_d) is the set of all first elements in the ordered pairs of R and the Range of R (denoted R_r) is the set of all second elements in the ordered pairs of R.

Example #3: Let $W = \{(2,5);(9,-3);(-6,6);(0,-3)\}$ be a relation. State the domain and range of W.

Answer: $W_d = \{2,\ 9, -6,\ 0\}$ and $W_r = \{5,-3,6\}$

Note: Repeated values in the relation are written only once in either the domain or range and, as with any set, the order of set elements is not necessarily important.

Now that we have examined the basics of a relation, we are ready to define one of the most useful and important concepts in higher mathematics; the Function.

Function

Given a relation $R \in \mathbb{R}$, then $R = F$ is a 'Function' $F \in \mathbb{R}$ if for any element of the Domain $(a_n \in R_d)$, there is one and only one element in the Range $(b_n \in R_r)$.

Example #4: Let $R = \{(2,5);(-3,0);(5,6)\}$, then $R = F$ is a function since each element of the domain, $R_d = \{2,-3,\ 5\}$, pairs with only one element of the range $R_r = \{5,0,6\}$.

This can be seen in the following diagram:

$$\begin{array}{cc} R_d & R_r \\ 2 \rightarrow 5 \\ -3 \rightarrow 0 \\ 5 \rightarrow 6 \end{array}$$

Example #5: Let $W = \{(5,9);(6,-2);(-5,3);(6,0)\}$, then $W \neq F$, is not a function since at least one element of the domain, 6, pairs with more than one element of the range, -2 and 0.

This can be seen the following diagram:

$$\begin{array}{cc} W_d & \mathrm{W}_r \\ 5 \rightarrow 9 \\ -5 \rightarrow 3 \\ 6 \rightarrow -2, 0 \end{array}$$

Two different values $(-2,\ \text{and } 0)$ in the range of W, are paired with the single value of '6' in the domain.

Example #6: Let $Q = \{(5,-9);(6,8);(2,-2);(-7,-9)\}$, then $Q = F$, is a function since every element in the domain, , pairs with only one element in the range, Q_r, even though the value, -9, is repeated in the range.

This can be seen the following diagram:

$$\begin{array}{cc} Q_d & Q_r \\ 2 \rightarrow & -2 \\ 6 \rightarrow & 8 \\ -7 \searrow & \\ & -9 \\ 5 \nearrow & \end{array}$$

Although the values of 5 and -7, in Q_d, each pair with -9 in the Q_r, neither 5 nor -7 pair with any other value in Q_r. This satisfies the definition of function, which states that, "…for any element of the domain, there is one and only one element in the range".

Notation: If a relation is a function, then the following notation is commonly used:

Let $R = F$ be a function, then "$f(x) = F$". This is read as," f of x equals the function F".

Note: The $(\ )$'s in the notation do not mean to multiply f times x.

In mathematics it is common for one form of notation to be used to represent two or more different concepts.

Examples of Variable Functions

For the most part, this course will focus on functions whose domain is a variable quantity. The following is a sample list of 'variable functions'.

1.) $f(x) = x^2 + 3x - 8$ — A **Polynomial** function.

2.) $C(r) = 2\pi r$ — The **Circumference of a Circle** function with variable radius.

3.) $I(t) = prt$ — The **Interest Rate** function for any amount of time, t, given an amount of money (p), and a rate of interest (r).

4.) $g(x) = 3^x$ — An **Exponential** function calculating variable powers of 3.

In the above examples, the variable in the $(\ )$'s represents all values of the domain that can be used in the function to calculate a result or, "range value" of the function at that given point.

Example #7: Calculate the range values for the Circumference of a Circle function given $C_d = \{2,\ 10,\ 0,\ -3\}$.

Answer: For this problem the values, $\{2,\ 10,\ 0, -3\}$ will be substituted individually for the variable r in, $C(r) = 2\pi r$. This is performed in the following manner:

1.) If $r = 2$, then $C(2) = 2\pi 2 = 4\pi \approx 12.566$
2.) If $r = 10$, then $C(10) = 2\pi 10 = 20\pi \approx 62.832$
3.) If $r = 0$, then $C(0) = 2\pi 0 = 0$
4.) If $r = -3$, then $C(-3) = 2\pi(-3) = \varnothing$

Note: the radius of a circle must always be ≥ 0 because the radius represents length, which must always be positive.

Notice in the above example that in $C(r)$ the 'r' acts a placeholder for successive domain values that will be used to evaluate the function. Also notice that for $r = 0$, $C(0) = 0$. Any domain value that causes the function value to equal 0 is called a '**zero**' or '**root**' of the function.

Example #8: Determine the 'zeros' of the function;

$$h(x) = x^2 - 7x + 6 \text{ for } h_d = \{-3,\ 0,\ 1,\ 4,\ 6\}$$

Answer:

$$h(-3) = (-3)^2 - 7(-3) + 6 = 9 + 21 + 6 = 36 \neq 0$$

$$h(0) = (0)^2 - 7(0) + 6 = 0 - 0 + 6 = 6 \neq 0$$

$$h(1) = (1)^2 - 7(1) + 6 = 1 - 7 + 6 = 0: \text{ (root of } h(x))$$

$$h(4) = (4)^2 - 7(4) + 6 = 16 - 28 + 6 = -6 \neq 0$$

$$h(6) = (6)^2 - 7(6) + 6 = 36 - 42 + 6 = 0: \text{ (root of } h(x))$$

Therefore: $h(1) = 0$ and $h(6) = 0$, and $x = 1$ and $x = 6$ are, 'zeros' of the function.

Miscellaneous Examples

1.) Given: $g(x) = 3x - 8$, find $g(2)$

Answer: $g(2) = 3(2) - 8 = 6 - 8 = -2$

2.) Given: $p(x) = \frac{6}{x}$, where $x \neq 0$; find $p\left(\frac{1}{3}\right)$

Answer: $p\left(\frac{1}{3}\right) = \frac{6}{\frac{1}{3}} = 6 \div \frac{1}{3} = 6 \times 3 = 18$

3.) Given: $w(x) = 10x^2 - 19x + \frac{1}{x}$, where $x \neq 0$; find $w(k)$:

Answer $w(k) = 10k^2 \text{-} 19k + \frac{1}{k}$, where $k \neq 0$

4.) Given: $f(x) = 5x - 9$, find $f(y+z)$

Answer: $f(y+z) = 5(y+z) - 9 = 5y + 5z - 9$

5.) Given $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$, find $s(4)$

Answer $s(4) = \frac{1}{2}a4^2 + v_0 4 + s_0 = \frac{1}{2} \times 16a + 4v_0 + s_0 = 8a + 4v_0 + s_0$

The Algebra of Functions

Although relations and functions may be written as sets, for the most part we will examine functions that provide relationships between two or more real variables in algebraic equations. Later we will examine exponential, logarithmic, trigonometric, and special functions throughout this course.

Algebraic Properties of Functions

Let $f(x),\ g(x) \in \mathbb{R}$ be real-valued functions and denoted by $f = f(x),\ \ g = g(x)$, then all the following properties hold:

1.) Addition Property	$f + g = f(x) + g(x)$
2.) Subtraction Property	$f - g = f(x) - g(x)$
3.) Multiplication Property	$f \times g = f(x) \times g(x)$
4.) Division Property:	$f \div g = f(x) \div g(x);\ \text{for } g(x) \neq 0$
5.) Composition of Functions*	$f \circ g = f(g(x))$ *

*This property will be discussed in detail shortly and in the next unit.

Example #1: Let $f(x) = 3x - 5,\ \ g(x) = x^2 + 3x + 2,\ h(x) = \dfrac{3}{x+2}$,

$k(x) = -9,\ \ p(x) = 9x^2 - 25$

1.) Find: $(f + g)(x)$

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= (3x-5) + (x^2 + 3x + 2)\\ &= 3x - 5 + x^2 + 3x + 2\\ &= x^2 + 6x - 3\end{aligned}$$

2.) Find: $(h \times k)(x)$

$$\begin{aligned}(h \times k)(x) &= h(x) \times k(x)\\ &= \left(\frac{3}{x+2}\right) \times (-9)\\ &= \frac{-27}{x+2}\end{aligned}$$

3.) Find: $(g-h)(x)$

$$(g-h)(x) = g(x) - h(x)$$
$$= (x^2 + 3x + 2) - \left(\frac{3}{x+2}\right)$$
$$= x^2 + 3x + 2 - \frac{3}{x+2}$$

(Simplification as one fraction is not always necessary)

4.) Find: $\left(\frac{f}{p}\right)(x)$

$$\left(\frac{f}{p}\right)(x) = \frac{f(x)}{p(x)}$$
$$= \frac{(3x-5)}{(9x^2-25)}$$
$$= \frac{(3x-5)}{(3x-5)(3x+5)}$$
$$= \frac{1}{(3x+5)} \quad x \neq -\frac{5}{3}, \frac{5}{3}$$

(Simplification by factoring is common practice, **but** the domain is determined before factoring.)

5.) Find: $f(k(x))$ The Composition of Functions*

We know $k(x) = -9$, therefore

$$f(k(x)) = f(-9)$$
$$= 3(-9) - 5$$
$$= -27 - 5$$
$$= -32$$

6.) Find: $f(g(x))$ The Composition of Functions*

$g(x) = x^2 + 3x + 2$, in $f(x)$, ' x ' is seen to be a 'placeholder' for all domain or variable values to be used in the function. Therefore all of $g(x)$ is now the domain of $f(x)$, or,

$$\begin{aligned} f(g(x)) &= f(x^2 + 3x + 2) \\ &= 3(x^2 + 3x + 2) - 5 \\ &= 3x^2 + 9x + 6 - 5 \\ &= 3x^2 + 9x + 1 \end{aligned}$$

7.) Find: $h(f(x))$ The Composition of Functions*

Now all of $f(x)$ is now the domain of $h(x)$, or,

$$\begin{aligned} h(f(x)) &= h(3x - 5) \\ &= \frac{3}{(3x-5)+2} \\ &= \frac{3}{3x-3} \\ &= \frac{3}{3(x-1)} \\ &= \frac{1}{(x-1)} \quad x \neq 1 \end{aligned}$$

The Composition of Functions will be examined more at a later time.