SETS & INTERVALS

In this unit you will learn basic operations, definitions, and notation for “sets”. In the second part of the unit we will explore the various types of number intervals as sets and develop an understanding of how these two topics define and set parameters for higher mathematical explorations.

Definitions and Notation

Set: Any well-defined collection of objects that share a common property.

Notation: Sets are generally denoted by upper case letters printed with the objects within the set encased in $\{\ \}$'s.

Example #1: Let $A = \{2, 4, 6, 8\}$ and $B = \{\mathbb{Q}$, dog, corn, MacBeth$\}$

The above sets are properly read as:

"*A* is equal to the set of elements 2, 4, 6, 8."
"*B* is equal to the set of elements $\mathbb{Q}$, dog, corn, MacBeth."

Notice that the list of elements for set B does not appear to share a common property; however, in Set Theory, if a list of items (elements) shares nothing, then those elements constitute a set of elements that share "nothing" as their common property.

-The objects in a set are called **set elements**.

When any one or group of elements of a set is particularly identified, we use the following notation:

Let $W = \{1, 4, 9, 16, 25\}$ and $R = \{2, 4, 6, 96, 98, 100\}$,

then $9 \in W$ and $6, 98 \in R$.

This is properly read as,"9 is an element of *W*" and, "6 and 98 are elements of *R*".

Special Sets

Closed Set

*1) **Closed Set:** The sets *A*, *B*, *R* and *W* from above are considered being 'closed sets'. Closed sets are sets that have all elements listed with no indication that other elements are also in the set but not listed. For instance, the set *A* above lists the first four positive even whole numbers, (2, 4, 6, 8) and only those four. Although other even numbers exist, only the first four are listed with no indication that others may be included. For set *B*, there is no indication that the set might also include other letters than $\mathbb{Q}$, other animals than 'dog', other vegetables besides 'corn', or other Shakespearean plays besides 'MacBeth'.

-If other elements are included as part of a set, they would somehow be indicated such as:

$$R = \{2, 4, 6, ..., 96, 98, 100\}$$

Now set *R* can be understood to contain all even numbers between 2 and 100 inclusive; however *R* is still a closed set since it begins with 2 and ends with 100.

Open Set

*2) **Open Set:** Open sets fall into two categories:

a) Open
b) Half-Open

Suppose set *R* above is now written thusly:

$$R = \{2, 4, 6, 8, ...\}$$

Set *R* is now called **'Half-Open'**. The first element in *R* is '2', but the three dots after '8' indicate that *R* also contains all positive even numbers without end. A half-open set will either contain a 1st element and no last element, or it will not contain a 1st element but will contain a last element such as:

$$R = \{..., -4, \ -2, \ 0, \ 2, \ 4, \ 6, \ ..., \ 96, \ 98, \ 100\}$$

Set *R* now contains all positive even numbers up to and including 100, also contains all negative even numbers to infinity, and includes '0'.

If now we write set *R* as:

$$R = \{..., \ -4, \ -2, 0, 2, 4, 6, 8, ...\}$$

Set R now includes all even numbers from negative to positive infinity and zero. Since R lists no 1st or last element, we say that set A is now "fully open" or simply **'Open'**.

Empty Set

*3) **Empty Set:** The 'Empty Set' is any set that contains no elements. The empty set is commonly denoted in one of two ways;

$Q = \{\ \}$ or $Q = \varnothing$

Note: $Q = \{\varnothing\}$ is not an empty set since the set Q 'contains' the empty set inside it as an element.

Sample Problems: Identify the following descriptions as constituting 'Closed', 'Open', 'Half-Open', or 'Empty, $\varnothing$' Sets:

(1) All even prime numbers.

Answer: Closed Set

Since 2 is the only even prime #, no other elements can be included in this set.

(2) All perfect squares between 50 and 60.

Answer: Empty Set represented by $\varnothing$

Since $7^2 = 49$ and $8^2 = 64$, no perfect squares are found between 50 and 60.

(3) All women born between January 1, 2004 and December 31, 3000.

Answer: Half-Open

The 1st female birth has already occurred so this set contains a 1st element, but the last birth is not yet known, so the set remains half-open. On January 1, 3001 the set will be closed.

(4) The list of all rectangles whose area is 30 square meters.

Answer: Open

There are an infinite number of rectangles whose area equals 30 square meters if we include fractions of a meter for the lengths and widths.

(i.e., let $W = \frac{60}{7}$ and $L = \frac{7}{2}$, then $L*W = \frac{60}{7}*\frac{7}{2} = \frac{60}{2} = 30$).

In such a situation no first or last rectangle can be identified.

Subset: Any collection of the elements of a set.

The key to this definition is that any collection of the elements of the original set constitutes a subset of that set. Therefore, the original set is always a subset of itself and the empty set is always a subset of any set since it represents a collection of no elements of the original set.

Example #1:

Let $M = \{\sqrt{2}, \pi, \frac{21}{31}\}$ and $N = \{\sqrt{2}, \frac{21}{31}\}$,

then N is a subset of M and is denoted "$N \subset M$"

Example #2:

Let $R = \{1, 2, 3\}$. List all the subsets of R.

Answer: {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}, and $\varnothing$

Notice that set R contains a total of three elements and the total number of subsets of $R = 8$ which is $2^3 = 8$.

Theorem #1: The number of subsets of a set with 'n' elements is given by '$P = 2^n$'and is called the "Power Set" of the original set.

Example #3:

Let $F = \{\text{red, yellow, green, blue}\}$. How many subsets of F can be formed from F?

Answer: $n = 4$ elements, therefore there are $2^4 = 16$ subsets of F.

Operations on Sets

Intersection: Given two or more sets, *A* and *B*, the intersection of *A* and *B* is the set whose elements are common to both *A* and *B*.

Notation: The intersection of two sets is given by '$A \cap B$'.

Union: Given two or more sets, *A* and *B*, the union of *A* and *B* is the set that combines all elements of *A* with all elements of *B*.

Notation: The union of two sets is given by '$A \cup B$'.

Example #1: Let A={1, 2, 3, 4, 5}; B={2, 4, 6, 8}

Find $A \cap B$: Since an intersection lists all common elements between the sets then

$A \cap B$={2, 4}

Find $A \cup B$: Since a union combines all elements in both sets then

$A \cup B$={1, 2, 3, 4, 5, 6, 8}

Example #2: Let P={5, 8, 9}; Q={6, 7, 10}

Find $A \cap B$: Since *A* and *B* share no common elements between then

$A \cap B$= { } or $\varnothing$

Find $A \cup B$: Nearly any two sets can combine all elements therefore

$A \cup B$= {5, 6, 7, 8, 9, 10}

For set operations on more than two sets, only one operation may occur at a time. In these cases we indicate which operation is to occur 1^{st} by ( )'s.

Example #3: Let X={-3, 5, 12} , Y={0, 12, 17} , and Z={-4, -3, 0, 5, 17}

Find $(X \cap Y) \cup Z$:

Step #1: Find $X \cap Y$={12}
Step #2: Find {12} $\cup$ Z={−4, −3, 0, 5, 12, 17}

Find $(X \cup Z) \cap Y$:

Step #1: Find $X \cup Z = \{-4, -3, 0, 5, 12, 17\}$
Step #2: Find $\{-4, -3, 0, 5, 12, 17\} \cap Y = \{0, 12, 17\}$

Find $(X \cap Y) \cap Z$:

Step #1: Find $X \cap Y = \{12\}$
Step #2: Find $\{12\} \cap Z = \{\ \}$

Associated with an intersection of sets is the word "**and**".

Associated with a union of sets is the word "**or**".

Let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, then the following are properly read as:

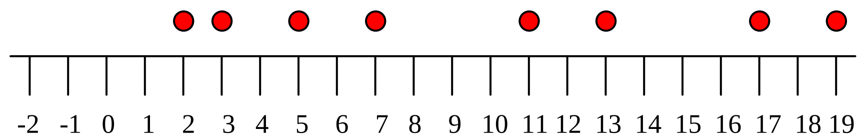
$A \cap B = \{2\}$ *is read* "The intersection of *A* '**and**' *B* equals 2".
$A \cup B = \{1, 2, 3, 4\}$ *is read* "The union of *A* '**or**' *B* equals 1, 2, 3, 4".

Intervals on the Number Line

As noted, a set may contain any collection of elements whether they are mathematical or otherwise. In mathematics a set may contain geometric, algebraic, numerical or any collection of the elements of mathematics. When a mathematical set contains numerical information, we examine that set as containing elements from the number line. Number line sets are called, **"Interval Sets".**

Continuous and Discontinuous Intervals

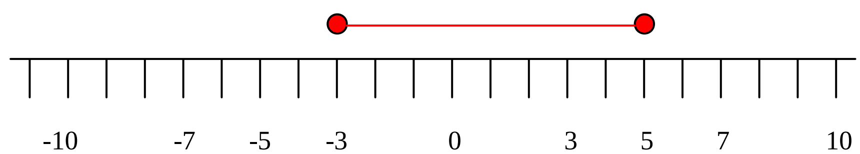
Let $A = \{\text{All prime \#'s}\}$. Since not all numbers are prime numbers, we say that set A represents a 'discontinuous interval' on the real number line. This can be seen graphically in the following way:



By marking each prime # with a filled in red dot, it is clearly seen that many numbers are not included in set *A*. Therefore there is no continuity for set *A* (the set of all prime numbers) and *A* is a discontinuous interval.

Let $B = \{\text{all numbers greater than or equal to} -3 \text{ and less than or equal to } 5\}$.

This is a continuous interval and can also be seen graphically as:



For the most part, this course will deal with continuous intervals.

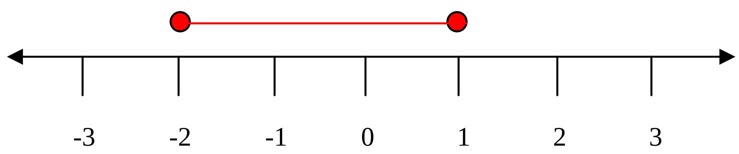
Types of Intervals and their Notation

Since an interval of numbers from the number line is also a set of numbers, we can say that, like a set, an interval can have one of the following descriptions that we also used with sets: a) Closed Interval, b) Open Interval, c) Half-Open Interval.

Note: There is no 'empty' interval. The entire number line is considered to be unbroken and continuous from negative to positive infinity and therefore nowhere empty. An interval with no entries is an interval of 'no width'.

*a) **Closed Intervals:** A closed interval from the number line is a continuous set of numbers with distinct endpoints.

Example #1:

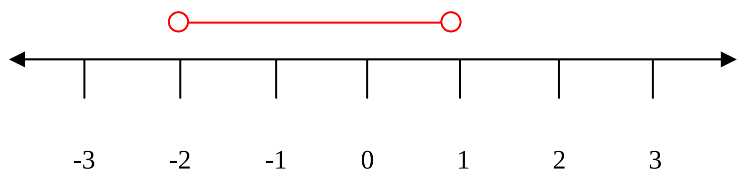


The graph above represents a closed interval from –2 to 1. The filled in dots above –2 and 1 indicate that these two numbers and all numbers in between are 'on' the interval. The following denotes this:

$A = [-2,1]$ The square brackets around –2 and 1 indicate that the interval is closed and that –2 and 1 are part of the interval. The comma indicates the interval is continuous.

*b) **Open Intervals:** An open interval from the number line is a continuous set of numbers but excludes the endpoints of the interval.

Example #2:

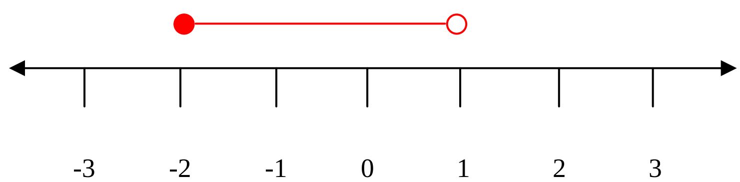


The above graph represents an open interval from –2 to 1. The open circles above –2 and 1 indicate that the interval consists of all numbers between –2 and 1, but the numbers themselves are not on the interval. In this case, –2 and 1 serve as boundaries for the interval. An open interval is denoted in the following way:

$A = (-2,1)$ The parentheses around –2 and 1 indicate that the interval is open and that –2 and 1 are not part of the interval. The comma indicates the interval is continuous.

*c) **Half-Open Intervals:** A half-open interval from the number line includes all numbers between two numbers, includes one endpoint on the interval, but excludes the other endpoint from the interval.

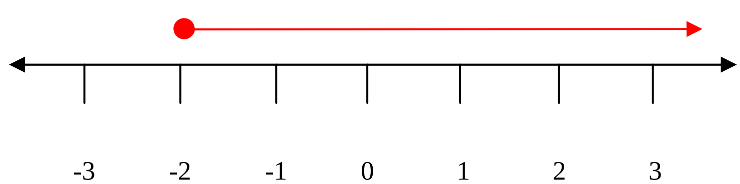
Example #3:



The above graph represents a half-open interval from –2 to 1. The filled in circle above –2 and the open circle above 1 indicate that the interval consists of all numbers between –2 and 1, but only the number –2 is also on the interval while 1 is excluded from the interval. A half-open interval is denoted in the following way:

$A = [-2,1)$ The bracket around –2 and parenthesis around 1 indicate that the interval is half-open and includes –2. The comma indicates the interval is continuous. If the interval was denoted by (–2,1] then –2 would be excluded, 1 would be included and still half-open.

Example #4:



The above graph represents a second form of half-open interval. The filled in circle above –2 indicates that 2 is on the interval and consists of all numbers from –2 to positive infinity. This type of half-open interval is denoted in the following way;

$$A = [-2,\ \infty)$$

We will explore intervals again and examine their use as solution sets to equations and inequalities later. First we will examine number types and their classification in mathematics.