COMPLEX NUMBERS

$a+bi$

Unit Overview

Imaginary and complex numbers are introduced in this unit. Complex numbers are extended to include the imaginary solutions of quadratic equations. The operations of complex numbers are examined including rationalizing the denominator of fractions that have radicals in the denominator.

The Discriminant

The **discriminant** of a quadratic equation gives an idea of the number of roots and the nature of the roots of an equation. If $ax^2+bx+c=0$ is the equation, then the discriminant of the equation is b^2-4ac.

The discriminant can be seen in the quadratic formula. The discriminant is the expression under the radical, b^2-4ac.

In the quadratic formula $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, the expression b^2-4ac is known as the **discriminant** and will identify how many and what type of solutions there are to a quadratic equation.

Types of Solutions

If the value of b^2-4ac is **positive**	2 real solutions
If the value of b^2-4ac is **negative**	2 imaginary solutions*
If the value of b^2-4ac is **zero**	1 real solution

*Imaginary solutions are numbers that involve the square root of a negative number. The square root of a negative number is *undefined* in the "real" number system. In the next section, we will extend the number system to include complex numbers which have imaginary parts.

Example #1: Find the discriminant and determine the number of solutions for each of the quadratics shown below.

1.) $3x^2 - 6x + 4 = 0$	2.) $4x^2 - 20x + 25 = 0$	*3.) $9x^2 + 12x = -2$
$b^2 - 4ac$	$b^2 - 4ac$	$b^2 - 4ac$
$(-6)^2 - 4(3)(4)$	$(-20)^2 - 4(4)(25)$	$(12)^2 - 4(9)(2)$
$36 - 48 = -12$	$400 - 400 = 0$	$144 - 72 = 72$
2 imaginary solutions	1 real solution	2 real solutions

*Note: In the third quadratic equation, express the quadratic equation in standard form, $9x^2 + 12x + 2 = 0$, to determine a = 9, b = 12, and c = 2.

QuickTime The Quadratic Formula (06:38)

Stop! **Go to Questions #1-3 about this section, then return to continue on to the next section.**

Imaginary Numbers

In dealing with the quadratic formula, sometimes the discriminant is negative. In this case, we can simplify the formula further if we know about imaginary and complex numbers.

An imaginary number is a number in the form of ai where a is any real number and $i^2 = -1$; thus $i = \sqrt{-1}$.

> If $r > 0$, then the **imaginary number** $\sqrt{-r}$ is defined as the following
>
> $$\sqrt{-r} = \sqrt{-1 \cdot r} = \sqrt{-1} \cdot \sqrt{r} = i\sqrt{r}$$

When finding square root of a negative number, the first thing to do is factor out a (–1) which is equal to *i*.

Example #1: Simplify: $\sqrt{-25}$

$\sqrt{-25} =$

$= \sqrt{-1 \cdot 25}$	$-25 = (-1)(25)$
$= \sqrt{-1} \cdot \sqrt{25}$	The product of two numbers under a radical can be expressed as the product of two radicals.
$= i \cdot 5$	$i = \sqrt{-1}$
$= 5i$	*Apply the Commutative Property of Multiplication.

$\sqrt{-25} = 5i$

*Note: Always put the number value first unless the expression contains a radical sign.

Imaginary and Complex Numbers -- Electricity (04:35)

Example #2 : Simplify: $\sqrt{-21}$

$\sqrt{-21}$	$=$	
	$= \sqrt{-1 \cdot 21}$	$-21 = (-1)(21)$
	$= \sqrt{-1} \cdot \sqrt{21}$	The product of two numbers under a radical can be expressed as the product of two radicals.
	$= i \cdot \sqrt{21}$	$i = \sqrt{-1}$
	$= i\sqrt{21}$	Simplify

$$\sqrt{-21} = i\sqrt{21}$$

Example #3 : Simplify: $\sqrt{-75}$

$\sqrt{-75}$	$=$	
	$= \sqrt{-1 \cdot 75}$	$-75 = (-1)(75)$
	$= \sqrt{-1} \cdot \sqrt{75}$	The product of two numbers under a radical can be expressed as the product of two radicals.
	$= i \cdot \sqrt{75}$	$i = \sqrt{-1}$
	$= i \cdot \sqrt{25 \cdot 3}$	25 is a perfect square.
	$= i\sqrt{25} \cdot \sqrt{3}$	The product of two numbers under a radical can be expressed as the product of two radicals.
	$= i \cdot 5 \cdot \sqrt{3}$	$\sqrt{25} = 5$
	$= 5i\sqrt{3}$	Commutative Property ($i \cdot 5 = 5 \cdot i$)

$$\sqrt{-75} = 5i\sqrt{3}$$

Cyclic Powers of *i*

The powers of *i* are cyclic and repeat in a pattern of 4 numbers. Thus, the powers of *i* have four possible outcomes.

$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$ (also i^0)

If we continue the pattern, we have the following:

$$i^5 = i^4 \cdot i = (1)i = i$$

$$i^6 = i^4 \cdot i^2 = (1)(-1) = -1$$

$$i^7 = i^4 \cdot i^3 = (1)(-i) = -i$$

$$i^8 = i^4 \cdot i^4 = (1)(1) = 1$$

The table below is extended to show the cyclic pattern.

$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$ (also i^0)
$i^5 = i$	$i^6 = -1$	$i^7 = -i$	$i^8 = 1$

Let's investigate this pattern further.

What are the values of i^9, i^{10}, i^{11}, and i^{12}?

Click here" to check your answer.

***i*, –1, – *i*, and 1**

When the powers of i equal one $(i^4 = 1,\ i^8 = 1,\ i^{12} = 1$ and so on$)$, the exponents of i are multiples of what number?

Click here" to check your answer.

Four

What is the value of i^{99}?

Click here" to check your answer.

$$\boldsymbol{i^{99} = i^{96} * i^3 = 1 * -i = -i}$$

Example #4: $i^{26} = ?$

$i^{26} = i^{24} \cdot i^{2}$	Write the exponents as a product of exponents.
$= 1 \cdot i^{2}$	$i^{24} = 1$ (The cylic pattern shows that exponents of *i* that are multiples of 4 are equal to 1.)
$= 1 \cdot (-1)$	$i^{2} = -1$
$= -1$	

$i^{26} = -1$

Stop! **Go to Questions #4-9 about this section, then return to continue on to the next section.**

Complex Numbers

> A **complex number** is any number that can be written as $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, a is called the **real part** and b is called the **imaginary part**.

*Two complex numbers are equal if the real parts are equal and the imaginary parts are equal.

Example #1: Solve for "x" and "y": $-3x + 4iy = 21 - 16i$

real parts	**imaginary parts**
$-3x = 21$	$4iy = -16i$
$x = -7$	$y = -4$

Thus $x = -7$ and $y = -4$

QuickTime Complex Numbers -- Electricity (03:15)

Now let's look at how imaginary numbers are used in finding the solutions to some quadratic equations with solutions that are expressed as complex numbers.

Example #2: Find the solution to the following quadratic: $x^2 - 4x + 13 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$, $b = -4$, and $c = 13$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$ Substitute.

$$x = \frac{4 \pm \sqrt{16 - 52}}{2}$$ Simplify.

$$x = \frac{4 \pm \sqrt{-36}}{2}$$ Simplify further.

$$x = \frac{4 \pm 6i}{2}$$ Express $\sqrt{-36}$ as an imaginary number.

$\left(\sqrt{-36} = \sqrt{-1 \cdot 36} = \sqrt{-1} \cdot \sqrt{36} = 6i\right)$

$$x = \frac{4 + 6i}{2} \quad \text{or } x = \frac{4 - 6i}{2}$$ Write the solutions as two answers.

Now, since the numerator of the fractions has a 2 common to both terms, we can further simplify the solutions by factoring out a 2 and cancelling.

$$x = \frac{2(2+3i)}{2} = \frac{\cancel{2}^{1}(2+3i)}{\cancel{2}^{1}} = 2 + 3i \quad \text{or} \quad x = \frac{2(2-3i)}{2} = 2 - 3i$$

The two simplified solutions are $2 + 3i$ or $2 - 3i$.

Example #3: Find the solution to the following quadratic: $6x^2 - 3x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ a = 6, b = –3, and c = 1

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(1)}}{2(6)}$$

$$x = \frac{3 \pm \sqrt{9 - 24}}{12}$$

$$x = \frac{3 \pm \sqrt{-15}}{12}$$

$$x = \frac{3 \pm i\sqrt{15}}{12}$$

Express $\sqrt{-15}$ as an imaginary number.

$\left(\sqrt{-15} = \sqrt{-1 \cdot 15} = \sqrt{-1} \cdot \sqrt{15} = i\sqrt{15}\right)$

$$x = \frac{3 + i\sqrt{15}}{12} \quad \text{or} \quad x = \frac{3 - i\sqrt{15}}{12}$$

The final answer may be simplified further. Since the numerator is the sum of two numbers, the fraction can be written as the sum of two fractions with the same denominator. This will then give us a fraction that can be simplified further.

$$x = \frac{3}{12} \pm \frac{i\sqrt{15}}{12}$$

$$x = \frac{1}{4} \pm \frac{i\sqrt{15}}{12}$$

The fraction $\frac{3}{12}$ reduces to $\frac{1}{4}$.

The two simplified solutions are $x = \frac{1}{4} + \frac{i\sqrt{15}}{12}$ or $\frac{1}{4} - \frac{i\sqrt{15}}{12}$.

Stop! **Go to Questions #10-13 about this section, then return to continue on to the next section.**

Computing with Complex Numbers

Adding and Subtracting Complex Numbers

To add or subtract complex numbers:
-combine the real parts
-combine the imaginary parts

Example #1: Find the sum: $(-10-6i)+(8-i)$

$(-10-6i)+(8-i)$	
$-10\ -6i\ +8\ -i$	Write without parenthesis.
$(-10+8)+(-6i-i)$	Rearrange the real and imaginary parts.
$-2+-7i$	Combine the real and imaginary parts.
$-2-7i$	Simplify

Example #2: Find the difference: $(-9+2i)-(3-4i)$

$(-9+2i)-(3-4i)$	
$-9+2i-3+4i$	Write without parenthesis.
	(Note $--4i=+4i$)
$-9-3+2i+4i$	Rearrange the real and imaginary parts.
$-12+6i$	Combine the real and imaginary parts.

Multiplying Complex Numbers

Multiplying two complex numbers is accomplished in a manner similar to multiplying two binomials. You can use the FOIL process of multiplication.

To multiply complex numbers in the form of a binomial times a binomial:
-use FOIL multiplication
-combine like terms
-change i^2 to (–1)

Example #3: Find the product: $(2-i)(-3-4i)$

$(2-i)(-3-4i)$	
$-6-8i+3i+4i^2$	FOIL
$-6-8i+3i+4(-1)$	Express $i^2=-1$
$-6-5i-4$	Simplify
$-10-5i$	Combine the real parts.

Example #4: Find the product: $(8-3i)(-4+5i)$

$(8-3i)(-4+5i)$	
$-32+40i+12i-15i^2$	FOIL
$-32+40i+12i-15(-1)$	Express $i^2=-1$
$-32+52i+15$	Simplify
$-17+52i$	Combine the real parts.

In the next example, an imaginary number is multiplied by a complex number by applying the distributive property.

Example #5: Find the product: $5i(-7-2i)$

$5i(-7-2i)$	
$5i(-7)-5i(2i)$	Distribute
$-35i-10i^2$	Simplify
$-35i-10(-1)$	Express $i^2=-1$
$-35i+10$	Simplify
$10-35i$	Apply the commutative property to express the complex number in proper form.

Stop! **Go to Questions #14-21 about this section, then return to continue on to the next section.**

Conjugate of a Complex Number

In order to simplify a fraction containing complex numbers, you often need to use the *conjugate of a complex number*. For example, the conjugate of $3+5i$ is $3-5i$ and the conjugate of $4-9i$ is $4+9i$.

> The conjugate of a complex number $a+bi$ is $a-bi$. The conjugate of $a-bi$ is $a+bi$.

To simplify a quotient with an imaginary number in the denominator, multiply by a fraction equal to 1, using the conjugate of the denominator. This is called **rationalizing the denominator**.

Example #1: Rationalize the denominator for the given complex number.

$$\frac{3}{-4+i}$$

The conjugate is $(-4-i)$.

$$\left(\frac{3}{-4+i}\right)\cdot\left(\frac{-4-i}{-4-i}\right)$$

Multiply the numerators, FOIL the denominators.

$$\frac{-12-3i}{16+4i-4i-i^2}$$

Combine like terms and simplify.

$$\frac{-12-3i}{16-(-1)}=\frac{-12-3i}{17}$$

$\frac{3}{-4+i}$ is expressed as $\frac{-12-3i}{17}$.

The denominator is rationalized.

(There is no i in the denominator.)

Example #2: Rationalize the denominator for the given complex number.

$$\frac{2+3i}{3-5i}$$ The conjugate is $3+5i$.

$$\left(\frac{2+3i}{3-5i}\right)\cdot\left(\frac{3+5i}{3+5i}\right)$$ FOIL the terms in the numerators and denominators.

$$\frac{6+10i+9i+15i^2}{9+15i-15i-25i^2}$$

$$\frac{6+19i+15(-1)}{9-25(-1)}$$ Simplify

$$\frac{-9+19i}{34}$$

$\frac{2+3i}{3-5i}$ is expressed as $\frac{-9+19i}{34}$. The denominator is rationalized. (There is no i in the denominator.)

Stop! **Go to Questions #22-29 to complete this unit.**