MATRICES

$$\begin{bmatrix} 3 & -9 & 4 \\ 0 & -6 & 3 \\ 5 & 13 & -4 \end{bmatrix}$$

Unit Overview

A matrix is a system of rows and columns that is used to organize numbers or data. In this unit you will learn how to add and subtract matrices, multiply a matrix by a constant and finally multiply two matrices.

Using Matrices to Represent Data

matrix: a system of rows and columns that is used as a tool for organizing numbers or data so that each position in the matrix has a purpose.

element: each value in a matrix, the numbers below

Example #1: $A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -4 \end{bmatrix}$ ← rows

↑ ↑ ↑
columns

*Matrices are named using their dimensions (rows × columns) therefore the matrix above would be known as a 2 × 3 matrix and would look something like this:

$$A_{2\times 3}$$

Each element of a matrix has a special location. For example –2 is in the first row, second column and would be represented as a_{12}, –4 would be represented as a_{23}.

In Matrix A, what number is located at a_{21}

"Click here" to check your answer.

The number in row two column one is 2.

Special Matrices

row matrix: only one row $\begin{bmatrix} 2 & 0 & -7 \end{bmatrix}$ *(1 × 3 matrix)

column matrix: only one column $\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$ *(3 × 1 matrix)

square matrix: same number of rows and columns

$\begin{bmatrix} 3 & 0 \\ -6 & 4 \end{bmatrix}$ or $\begin{bmatrix} 3 & -9 & 4 \\ 0 & -6 & 3 \\ 5 & 13 & -4 \end{bmatrix}$ *(2 × 2 matrix and 3 × 3 matrix)

Two matrices are considered equal if they have the same dimensions and each element of one matrix is equal to the corresponding element of the other.

Example #2: Solve for x and y. $\begin{bmatrix} 2x+y \\ x-3y \end{bmatrix} = \begin{bmatrix} 6 \\ 31 \end{bmatrix}$

*Since the matrices have the same dimensions and they are equal, the corresponding elements are equal. When you write the sentences that show this equality, two linear equations are formed. To solve for x and y use either substitution or elimination.

$$\begin{bmatrix} 2x+y \\ x-3y \end{bmatrix} = \begin{bmatrix} 6 \\ 31 \end{bmatrix}$$

$$\begin{aligned} 2x + y &= 6 \\ x - 3y &= 31 \end{aligned} \longrightarrow \begin{aligned} 3(2x + y &= 6) \\ x - 3y &= 31 \end{aligned} \longrightarrow \begin{aligned} 6x + 3y &= 18 \\ + \quad x - 3y &= 31 \\ \hline 7x &= 49 \\ x &= 7 \end{aligned}$$

Substitute 7 for x in either of the original equations to solve for y.

$$\begin{aligned} 2(7) + y &= 6 \\ 14 + y &= 6 \\ y &= -8 \end{aligned}$$

The solution to the system of equations is (7, –8).

Stop! **Go to Questions #1-7 about this section, then return to continue on to the next section.**

Adding or Subtracting Matrices

Matrices must have the same dimensions in order to add or subtract them. Combine the elements that have the corresponding location in the matrices.

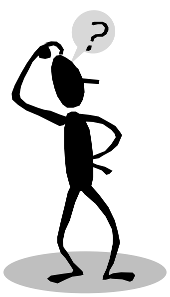
Let's take a look a few examples.

Example #1: Add the matrices: $\begin{bmatrix} 2 & -1 & 8 \\ 4 & 7 & 9 \end{bmatrix} + \begin{bmatrix} -1 & 4 & -3 \\ 7 & 2 & -6 \end{bmatrix} = ?$

$$\begin{bmatrix} \color{red}{2} & -1 & 8 \\ 4 & 7 & 9 \end{bmatrix} \color{red}{+} \begin{bmatrix} \color{red}{-1} & 4 & -3 \\ 7 & 2 & -6 \end{bmatrix} = \begin{bmatrix} \color{red}{2+-1} & -1+4 & 8+-3 \\ 4+7 & 7+2 & 9+-6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 11 & 9 & 3 \end{bmatrix}$$

Example #2: Add the matrices: $\begin{bmatrix} 5 & 7 & 3 \\ -1 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & -2 \\ 4 & 0 \end{bmatrix} = ?$



$$\begin{bmatrix} 5 & 7 & 3 \\ -1 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & -2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} & 5+1 & 7+0 & \color{red}{3+?} \\ + & -1+3 & 0+-2 & \color{red}{-4+?} \end{bmatrix}$$

This problem is not possible! The first matrix is a 2×3 and the second matrix is a 3×2.

To add or subtract matrices, what must be true about their size?

"Click here" to check your answer.

To add matrices the number of rows and columns must be the same.

Just like operations on real numbers, matrix addition follows some of the same properties.

Properties of Matrix Addition

If A, B, and C are $m \times n$ matrices, then

A + B is an $m \times n$ matrix	Closure Property of Addition
A + B = B + A	Commutative Property of Addition
(A + B) + C = A + (B + C)	Associative Property of Addition
A + O = O + A = A where O is a zero matrix	Additive Identity Property
For each A, there exists a unique opposite –A, such that A + (–A) = O	Additive Inverse Property

The **zero matrix**, or **additive identity** matrix is a matrix with all elements zero. Adding a zero matrix to a matrix leaves the matrix unchanged.

Example #3: Add: $\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = ?$

$$\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2+0 & 5+0 \\ -3+0 & 4+0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}$$

This is an example of the Additive Identity Property in matrix addition.

What is true about the elements of the additive identity matrix?

"Click here" to check your answer.

Each element is a zero.

The **opposite**, or **additive inverse** of a m x n matrix A is –A, where each element is the opposite of the corresponding element of A. When the two matrices are added, the result is the zero matrix.

Example #4: Add: $\begin{bmatrix} -2 & 3 & 5 \\ 0 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 & -5 \\ 0 & 1 & -4 \end{bmatrix} = ?$

$$\begin{bmatrix} -2 & 3 & 5 \\ 0 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 & -5 \\ 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is an example of the Additive Inverse Property in matrix addition.

What is true about the elements in the second matrix when comparing the elements in the same position of the first matrix?

"Click here" to check your answer.

Each element is the additive inverse of its counterpart in the first matrix.

Let's take a look at how to apply matrices to everyday problems.

Example #5: The employees at Kennedy's Bakery record the number of each type of cookie sold in the two stores for Monday, Tuesday and Wednesday. The sales are shown in the following table.

a) Write a matrix for each store's sales.

b) Find the sum of each type of cookie sold each day expressed as a matrix.

c) Find the difference in cookie sales from Store 1 to Store 2 expressed as a matrix.

Store 1			
	Mon	Tue	Wed
Chocolate chip	230	178	195
Sugar	198	200	184
Peanut Butter	115	126	98

Store 2			
	Mon	Tue	Wed
Chocolate chip	198	127	188
Sugar	127	163	151
Peanut Butter	93	88	76

a) Write a matrix for each store's sales.

$$\begin{bmatrix} 230 & 178 & 195 \\ 198 & 200 & 184 \\ 115 & 126 & 98 \end{bmatrix} \quad \begin{bmatrix} 198 & 127 & 188 \\ 127 & 163 & 151 \\ 93 & 88 & 76 \end{bmatrix}$$

b) Find the sum of each type of cookie sold each day expressed as a matrix.

Add the corresponding elements in each matrix.

$$\begin{bmatrix} 230 & 178 & 195 \\ 198 & 200 & 184 \\ 115 & 126 & 98 \end{bmatrix} + \begin{bmatrix} 198 & 127 & 188 \\ 127 & 163 & 151 \\ 93 & 88 & 76 \end{bmatrix} = \begin{bmatrix} 428 & 305 & 383 \\ 325 & 363 & 335 \\ 208 & 214 & 174 \end{bmatrix}$$

*Note: The sugar cookie (row 2) Wednesday sales (column 3) is highlighted in color (184 + 151 = 335).

c) Find the difference in cookie sales from Store 1 to Store 2 expressed as a matrix.

Subtract the corresponding elements in each matrix.

$$\begin{bmatrix} 230 & 178 & 195 \\ 198 & 200 & 184 \\ 115 & 126 & 98 \end{bmatrix} - \begin{bmatrix} 198 & 127 & 188 \\ 127 & 163 & 151 \\ 93 & 88 & 76 \end{bmatrix} = \begin{bmatrix} 32 & 51 & 7 \\ 71 & 37 & 33 \\ 22 & 38 & 22 \end{bmatrix}$$

*Note: The chocolate chip cookie (row 1) Tuesday sales (column 2) is highlighted in color (178 – 127 = 51).

Matrices provide a useful way to organize and calculate data.

QuickTime Tables without Labels -- Football (02:30)

Stop! **Go to Questions #8-13 about this section, then return to continue on to the next section.**

Scalar Multiplication

Multiplying a matrix by a constant, each element of the matrix is multiplied.

Example #1: Multiply matrix A by 3. (This is represented as 3A.)

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 6 & -4 \end{bmatrix} \qquad 3A = \begin{bmatrix} 6 & 3 \\ -9 & 0 \\ 18 & -12 \end{bmatrix}$$

*Notice that each element of matrix A was multiplied by three.

If scalar multiplication and addition or subtraction occurs in a problem, do the scalar multiplication first.

Example #2: Use scalar multiplication to simplify: $3\begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} - 5\begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} = ?$

$$3\begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} - 5\begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 21 \end{bmatrix} + \begin{bmatrix} 6 \\ -4 \\ 12 \end{bmatrix} + \begin{bmatrix} 10 \\ -15 \\ -30 \end{bmatrix} = \begin{bmatrix} 28 \\ -16 \\ 3 \end{bmatrix}$$

*Notice that the last matrix was multiplied by a (–5); therefore, it changes to addition (+ –5), and then all of the multiplications by –5 are within the matrix.

As in operations on real numbers, matrix scalar multiplication follows some of the same properties.

Properties of Scalar Multiplication

If A and B are $m \times n$ matrices, O is a zero matrix, and p and q are scalars.

pA is an $m \times n$ matrix	Closure Property
$pq(A) = p(qA)$	Associative Property
$p(A + B) = pA + pB$	Distributive Property
$(p + q)A = pA + qA$	Distributive Property
$1 \bullet A = A$	Identity Property

In the next example, we will explore the Associative Property of Scalar Multiplication.

Example #3: Show that 2(3)(A) = 2(3A) using scalar multiplication.

$$A = \begin{bmatrix} 4 & -2 \\ -6 & 3 \\ 0 & 5 \end{bmatrix}$$

Solution:

$$2(3)(A) = 2 \times 3 \times \begin{bmatrix} 4 & -2 \\ -6 & 3 \\ 0 & 5 \end{bmatrix} = 6\begin{bmatrix} 4 & -2 \\ -6 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 24 & -12 \\ -36 & 18 \\ 0 & 30 \end{bmatrix}$$

$$2(3A) = 2 \times \left(3\begin{bmatrix} 4 & -2 \\ -6 & 3 \\ 0 & 5 \end{bmatrix}\right) = 2 \times \begin{bmatrix} 12 & -6 \\ -18 & 9 \\ 0 & 15 \end{bmatrix} \times \begin{bmatrix} 24 & -12 \\ -36 & 18 \\ 0 & 30 \end{bmatrix}$$

Thus, 2(3)(A) = 2(3A), illustrating the associative property of scalar multiplication with matrices.

Stop! **Go to Questions #14-16 about this section, then return to continue on to the next section.**

Matrix Multiplication

Matrix multiplication involves multiplication and addition.

To multiply any two matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix.

Example #1: Multiply the matrices: $\begin{bmatrix} 5 & 4 & 2 \end{bmatrix} \times \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} = ?$

$$\begin{bmatrix} 5 & 4 & 2 \end{bmatrix} \times \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (5 \cdot 6) + (4 \cdot 1) + (2 \cdot 3) \end{bmatrix} = \begin{bmatrix} 40 \end{bmatrix}$$

*Notice that a 1 × 3 matrix multiplied by a 3 × 1 matrix results in a 1 × 1 matrix.

Matrix Multiplication

If matrix A has dimensions $m \times n$ and matrix B has dimensions $n \times r$, then the product AB has dimensions $m \times r$.

To multiply any two matrices,
(a) the *inner dimensions* must be the same,
(b) then the *outer dimensions* become the dimensions of the resulting product matrix.

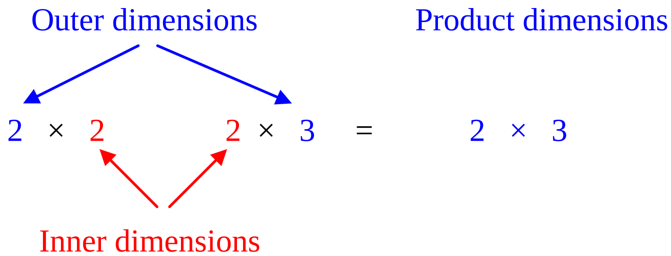
Outer dimensions　　　　Product dimensions

2 × 3　　3 × 4　=　2 × 4

Inner dimensions

Example #2: Multiply the matrices: $\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix} = ?$

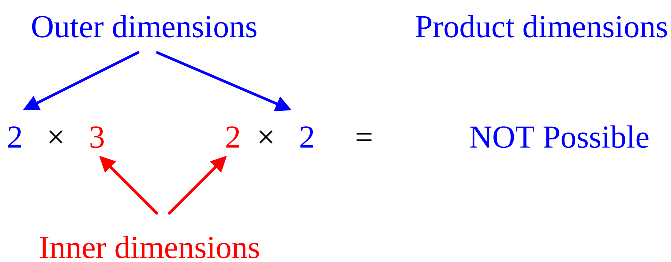
The first matrix is a 2 × 2 matrix and the second matrix is a 2 × 3 matrix.



$$\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 5(4)+3(0) & 5(2)+3(1) & 5(-1)+3(3) \\ 0(4)+1(0) & 0(2)+1(1) & 0(-1)+1(3) \end{bmatrix} = \begin{bmatrix} 20 & 13 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

* Notice that you are taking the first row [5   3] and multiplying each column, then picking up the second row and multiplying each column.

Example #3: Multiply the matrices: $\begin{bmatrix} 10 & 1 & 3 \\ -2 & 4 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix} = ?$



*Notice that if we attempted to multiply, there would be no number in the first column of the second matrix to multiply the 3 that is located in the first row of the first matrix.

$$\begin{bmatrix} 10 & 1 & 3 \\ -2 & 4 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 10(3)+1(-1)+3(?) & \\ & \end{bmatrix}$$

Since the **inner** dimensions are **not** the same, these two matrices cannot be multiplied. The number of columns in the first matrix must be the same as the number of rows in the second matrix.

Now let's take another look at how to apply matrices to everyday problems.

Example #4: The attendance for three basketball games is shown in the table below. Student tickets cost \$3.00 each and adult tickets cost \$5.00 each.

(a) Write matrices to represent the attendance and the ticket cost.
(b) Use matrix multiplication to show the revenue for ticket sales at each of the three games.
(c) Find the total revenue for the three games.

Ticket Sales		
	Students	Adults
Game 1	175	250
Game 2	200	320
Game 3	210	340

(a) Set up the matrices so that the number of columns in the attendance matrix is equal to the number of rows in the cost matrix.

Attendance Matrix × Cost Matrix = Total Ticket Revenue

$$\begin{bmatrix} 175 & 250 \\ 200 & 320 \\ 210 & 340 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

*In the cost matrix, include the cost of the student ticket (\$3) and the cost of the adult ticket (\$5). The cost of the student ticket is listed first because the number of ticket sales for student tickets is listed in the first column.

(b) Use matrix multiplication to show the revenue for ticket sales at each of the three games.

$$\begin{bmatrix} 175 & 250 \\ 200 & 320 \\ 210 & 340 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 175(3)+250(5) \\ 200(3)+320(5) \\ 210(3)+340(5) \end{bmatrix} = \begin{bmatrix} 1775 \\ 2200 \\ 2330 \end{bmatrix}$$

The revenue for Game 1 is \$1775.
The revenue for Game 2 is \$2200.
The revenue for Game 3 is \$2330.

(c) Find the total revenue for the three games.

$$1775 + 2200 + 2330 = 6305$$

The total revenue is \$6305.

QuickTime Matrix Arithmetic (03:27)

Multiplication of square ($n \times n$) matrices has some of the properties of real number multiplication.

Properties of Matrix Multiplication

If A, B and C are $n \times n$ matrices, and O is the $n \times n$ zero matrix

AB is an $n \times n$ matrix	Closure Property
(AB)C = A(BC)	Associative Property
A(B + C) = AB + AC	Distributive Property
(B + C)A = BA + CA	Distributive Property
OA = AO = O	Multiplicative

In the next example, we will explore the Distributive Property of Matrix Multiplication.

Example #2: Show that A(B + C) = AB + AC.

$$A = \begin{bmatrix} 5 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 0 \\ -7 & 7 \end{bmatrix}$$

First, A(B + C)

$$= \begin{bmatrix} 5 & -2 \end{bmatrix}\left(\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ -7 & 7 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 5 & -2 \end{bmatrix}\begin{bmatrix} 10 & -1 \\ -5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 5(10) + -2(-5) \\ 5(-1) + -2(7) \end{bmatrix}$$

$$= \begin{bmatrix} 60 \\ -19 \end{bmatrix}$$

Second, AB + AC

$$= \begin{bmatrix} 5 & -2 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 5 & -2 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ -7 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 5(4) + -2(2) \\ 5(-1) + -2(0) \end{bmatrix} + \begin{bmatrix} 5(6) + -2(-7) \\ 5(0) + -2(7) \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ -5 \end{bmatrix} + \begin{bmatrix} 44 \\ -14 \end{bmatrix}$$

$$= \begin{bmatrix} 60 \\ -19 \end{bmatrix}$$

Thus, A(B + C) = AB + BC illustrating the distributive property of matrix multiplication.

Stop! **Go to Questions #17-30 to complete this unit.**