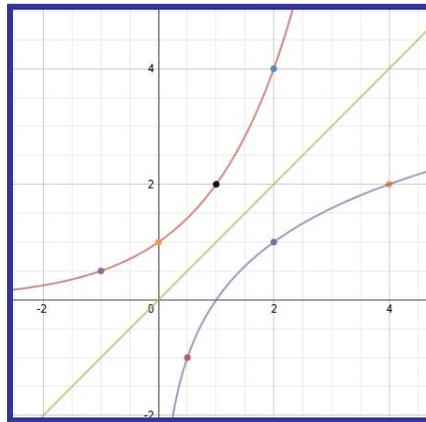


FUNCTIONS AND INVERSES OF FUNCTIONS



Unit Overview

The unit begins with basic function concepts: functions as relations, domain, range, and evaluating functions. The unit continues with performing operations with functions, finding the composition and inverse of functions, and determining whether the inverse of a function is a function.

Introduction to Functions

relation: a relationship between two variables such that each value of the first variable is paired with one or more values of the second variable; **a set of ordered pairs.**

Example #1: $\{(2, 4), (-4, 5), (2, -7), (0, 9)\}$

function: a relationship between two variables such that each value of the first is paired with exactly one value of the second variable; **all domain values (x-values) are different.**

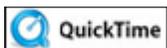
Example #2: $\{(2, 4), (0, 6), (7, 4), (-9, 4)\}$

domain: the set of all possible values of the first variable (all x -values)

From example #2 above: domain = $\{2, 0, 7, -9\}$

range: the set of all possible values of the second variable (all y -values)

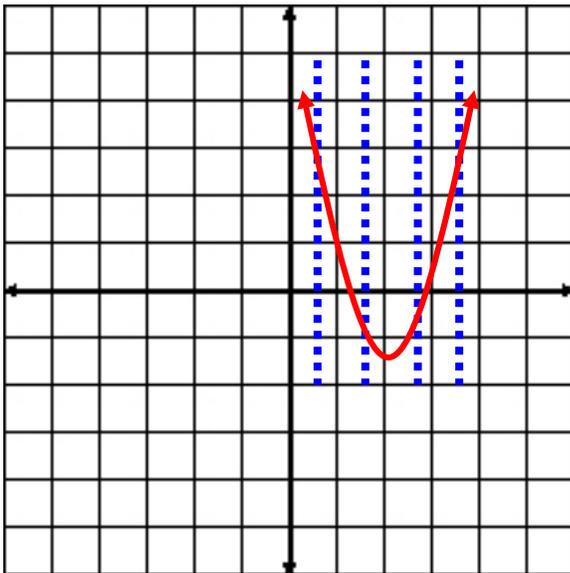
From example #2 above: range = $\{4, 6\}$



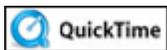
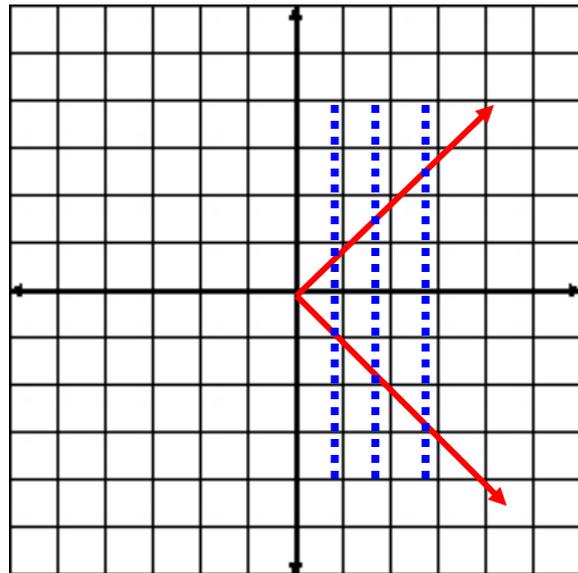
Introduction: Functions Relate Elements of Sets (02:04)

vertical line test: If every vertical line intersects a given graph at no more than one point, then the graph represents a function.

The vertical lines only intersect this graph at one point, therefore it is a function.



The vertical lines intersect the graph at more than one point; therefore it is **NOT** a function.



Functions, Domain, and Range -- Burning Calories (02:34)

function notation: If there is a correspondence between values of the domain, x , and values of the range, y , that is a function; then $y = f(x)$, and (x, y) can be written as $(x, f(x))$.

$f(x)$ is read “ f of x ”. The number represented by $f(x)$ is the value of the function f at x .

The variable x is called the **independent variable** and the variable y , or $f(x)$, is called the **dependent variable**.

To evaluate a function for a specific variable, replace x with the given value and solve.

Example #3: Evaluate $f(x) = -1.2x^2 + 4x - 3$ for $x = 1$.

$$f(x) = -1.2x^2 + 4x - 3$$

$$f(x) = -1.2(1)^2 + 4(1) - 3$$

$$f(x) = -1.2 + 4 - 3$$

$$f(x) = -0.2$$

When $x = 1$, the value of $f(x) = -0.2$

Example #4: Evaluate $f(x) = -1.2x^2 + 4x - 3$ for $x = 5$.

$$f(x) = -1.2x^2 + 4x - 3$$

$$f(x) = -1.2(5)^2 + 4(5) - 3$$

$$f(x) = -1.2(25) + 20 - 3$$

$$f(x) = -30 + 20 - 3$$

$$f(x) = -13$$

When $x = 5$, the value of $f(x) = -13$.

Example #5: Evaluate $g(x) = 3x^2 - x + 1$ for $x = 4$.

*Note: Other letters may be used when denoting functions.

$$g(x) = 3x^2 - x + 1$$

$$g(x) = 3(4)^2 - (4) + 1$$

$$g(x) = 3(16) - 4 + 1$$

$$g(x) = 48 - 4 + 1$$

$$g(x) = 45$$

When $x = 4$, the value of $g(x) = 45$.

Stop! Go to Questions #1-8 about this section, then return to continue on to the next section.

Operations with Functions

*functions can be combined by adding, subtracting, multiplying, and dividing.

Example #1: Let $f(x) = 4x^2 + 6x - 9$ and $g(x) = 6x^2 - x + 2$

Find $f + g$: $4x^2 + 6x - 9 + 6x^2 - x + 2$

$$4x^2 + 6x^2 + 6x - x - 9 + 2$$

$$f + g(x) = 10x^2 + 5x - 7$$

Find $f - g$: $4x^2 + 6x - 9 - (6x^2 - x + 2)$

$$4x^2 + 6x - 9 - 6x^2 + x - 2$$

$$4x^2 - 6x^2 + 6x + x - 9 - 2$$

$$f - g(x) = -2x^2 + 7x - 11$$

Example #2: Let $f(x) = 9x^2$ and $g(x) = 4x + 3$

Find $f \cdot g$: $9x^2(4x + 3)$

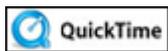
$$36x^3 + 27x^2$$

Example #3: Let $f(x) = 2x^2$ and $g(x) = x + 5$

Find $\frac{f}{g}$: $\frac{2x^2}{x+5}$ where $x \neq -5$ because that would make the denominator 0.

*To find restrictions on the domain, set the denominator equal to zero and solve. The result will be the restriction on the domain.

Composition of functions: when you apply a function rule on the result of another function rule, you **compose** the functions.



Composite Functions -- Barbeque (03:09)

Let f and g be functions of x .

The composition of f with g is denoted by $f \circ g$ or $f(g(x))$.

To find the value of a composite function:

-place the entire second function $f \circ (g(x))$ or $f(g(x))$ into the first function in place of x .

Example #4: Let $f(x) = x^2 - 1$ and $g(x) = 3x$

(a) Find $f \circ g(x)$ * place $g(x)$ into $f(x)$ for x

$$f(x) = x^2 - 1$$

$$f \circ g(x) = (3x)^2 - 1$$

$$f \circ g(x) = 9x^2 - 1$$

(b) Find $g \circ f(x)$ * place $f(x)$ into $g(x)$ for x

$$g(x) = 3x$$

$$g \circ f(x) = 3(x^2 - 1)$$

$$g \circ f(x) = 3x^2 - 3$$

Example #5: Let $f(x) = -2x^2 + 3$ and $g(x) = -4x$

Find $f(g(-1))$ * replace x in $g(x)$ with -1 ,
then replace that value into $f(x)$

$$g(x) = -4x$$

$$g(-1) = -4(-1)$$

$$g(-1) = 4$$

*determine $g(-1)$

$$f(g(-1)) = f(4)$$

$$f(x) = -2x^2 + 3$$

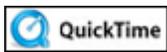
$$f(4) = -2(4)^2 + 3$$

$$= -2(16) + 3$$

$$= -32 + 3$$

$$= -29$$

*substitute 4 for $g(-1)$ in $f(x)$



Review of Functions and Polynomials (02:56)

Stop! Go to Questions #9-14 about this section, then return to continue on to the next section.

Inverses of Functions

The **inverse of a relation** consisting of the ordered pairs (x, y) is the set of all ordered pairs (y, x) .
(switch the x and y)

Consider the relation $\{(1, 2), (4, -2), (3, 2)\}$.

- The domain of the relation is $\{1, 4, 3\}$ and the range of the relation is $\{-2, 2\}$.
- The relation is a function because each domain value is paired with exactly one range value.

To find the **inverse** of the relation, switch the x any y values.

- The inverse is $\{(2, 1), (-2, 4), (2, 3)\}$.
- The domain of the inverse is $\{2, -2\}$.
- The range of the inverse is $\{1, 4, 3\}$.

*The relation is a function but the inverse is NOT a function because the domain value 2 is paired with two range values. $\{(2, 1), (-2, 4), (2, 3)\}$.

The range of a relation is the domain of the inverse. The domain of a relation is the range of the inverse. The inverse of a function may or may not be a function.

Let's consider the points in the table.

x	y
0	0
1	1
2	4
3	9

Is the relation a function?

Answer: The relation IS a function since each domain value (x) is paired with exactly one range value (y).

$$0 \rightarrow 0$$

$$1 \rightarrow 1$$

$$2 \rightarrow 4$$

$$3 \rightarrow 9$$

We denote the relation in function notation as $f(x)$ since it is a function.

x	$f(x)$
0	0
1	1
2	4
3	9

Is the inverse of $f(x)$ a function?

x	$f(x)$
0	0
1	1
2	4
3	9

Switch the domain and range.



x	$f^{-1}(x)$
0	0
1	1
4	2
9	3

* The inverse of a function f is denoted by f^{-1} .
This is read as “ f inverse” or “the inverse of f ”.

Answer: The inverse of the function IS a function also because each domain value is paired with exactly one range value.

$$0 \rightarrow 0$$

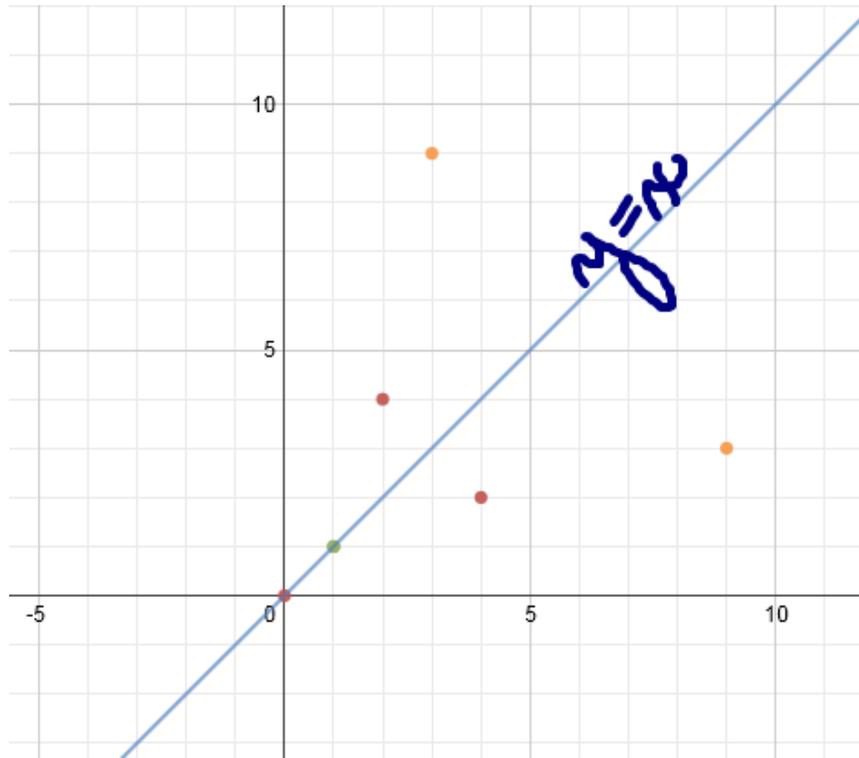
$$1 \rightarrow 1$$

$$4 \rightarrow 2$$

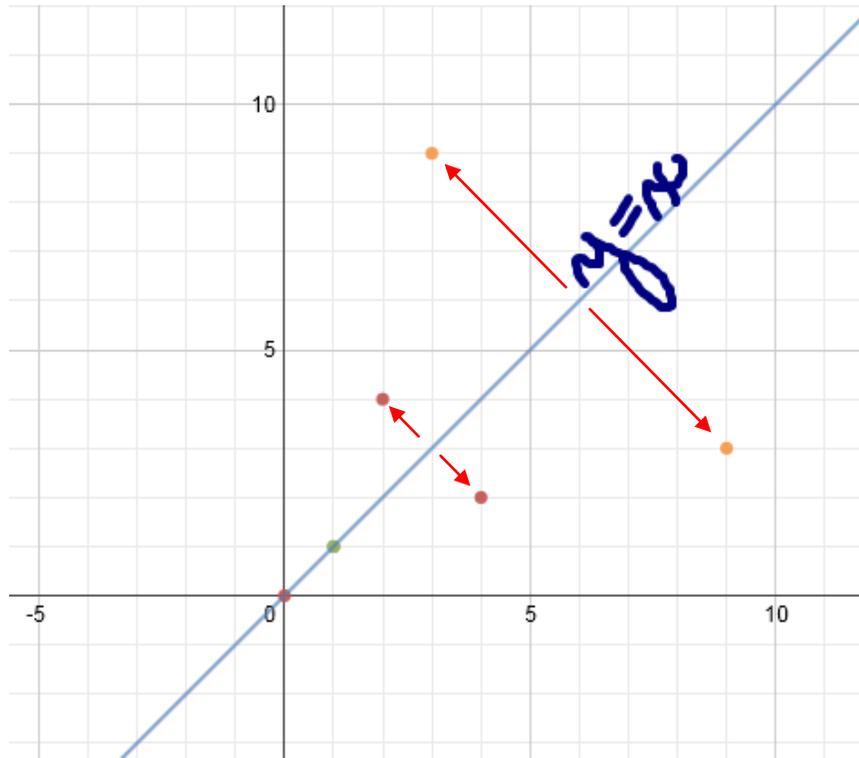
$$9 \rightarrow 3$$

Let's examine the graph of the function and its inverse, and how they relate to the graph of $y = x$.

Study the given graph of the points of the function, the inverse function, and $y = x$.



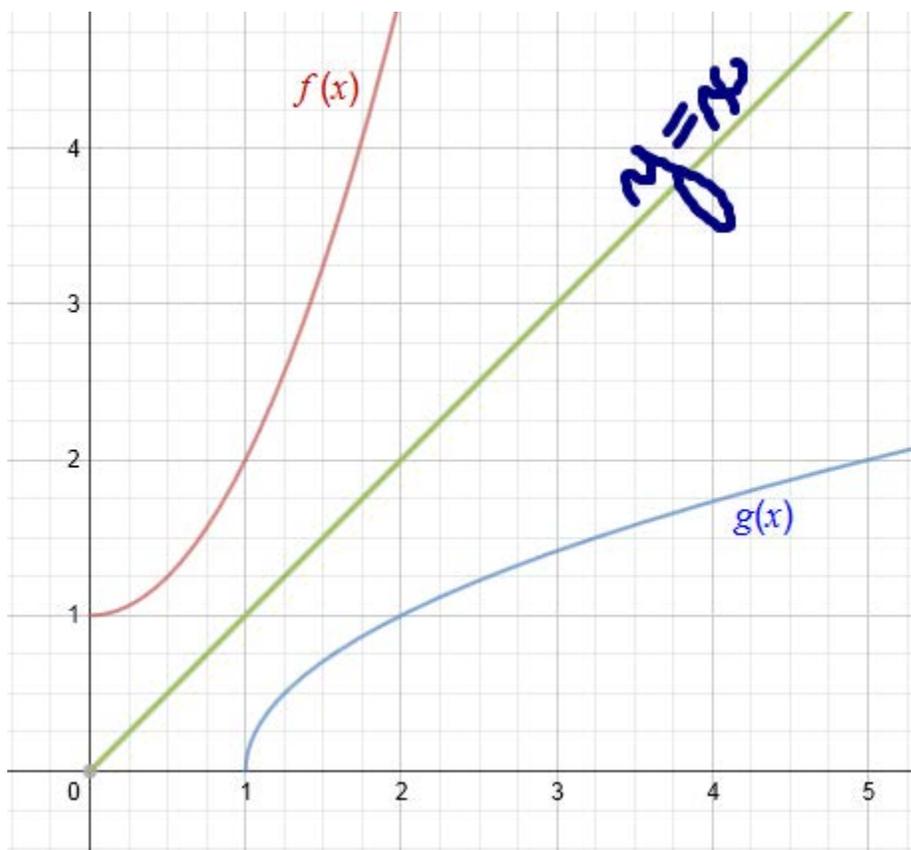
Do you see a relationship in the location of the points above $y = x$ (that is $f(x)$) and the location of the points below $y = x$, (that is the inverse of $f^{-1}(x)$)?



Notice that the points are a reflection of each over the graph of the line, $y = x$.

The graph of a function and its inverse are reflections of each other over the line $y = x$.

Study the graph of the two functions shown below.



Notice that all the points of $g(x)$ beginning at $(0, 0)$ are a reflection of the points in $f(x)$ across $y = x$; thus, $g(x)$ is the inverse of $f(x)$.

Since they are inverses, all of the domain values (x -values) in $f(x)$ are the range values (y -values) in $g(x)$ and all of the range values (y -values) in $f(x)$ are the domain values (x -values) in $g(x)$.

For example, notice that point $(1, 2)$ in $f(x)$ becomes point $(2, 1)$ on $g(x)$ when reflected across $y = x$.



What point on $g(x)$ is the inverse of point $(0, 1)$ on $f(x)$?

“Click here” to check your answer.

The reflected point is $(1, 0)$.



In general, any point (x, y) in $f(x)$ becomes what ordered pair in its inverse, $g(x)$?

“Click here” to check your answer.

(y, x)

If a function is defined by an equation, then the inverse of the function is found by switching the x and y in the equation, and then solving the new equation for y .

Example #1: $y = 3x - 2$
 $x = 3y - 2$

Switch the x and y .

$$x + 2 = 3y$$

Add 2 to each side.

$$\frac{x+2}{3} = y$$

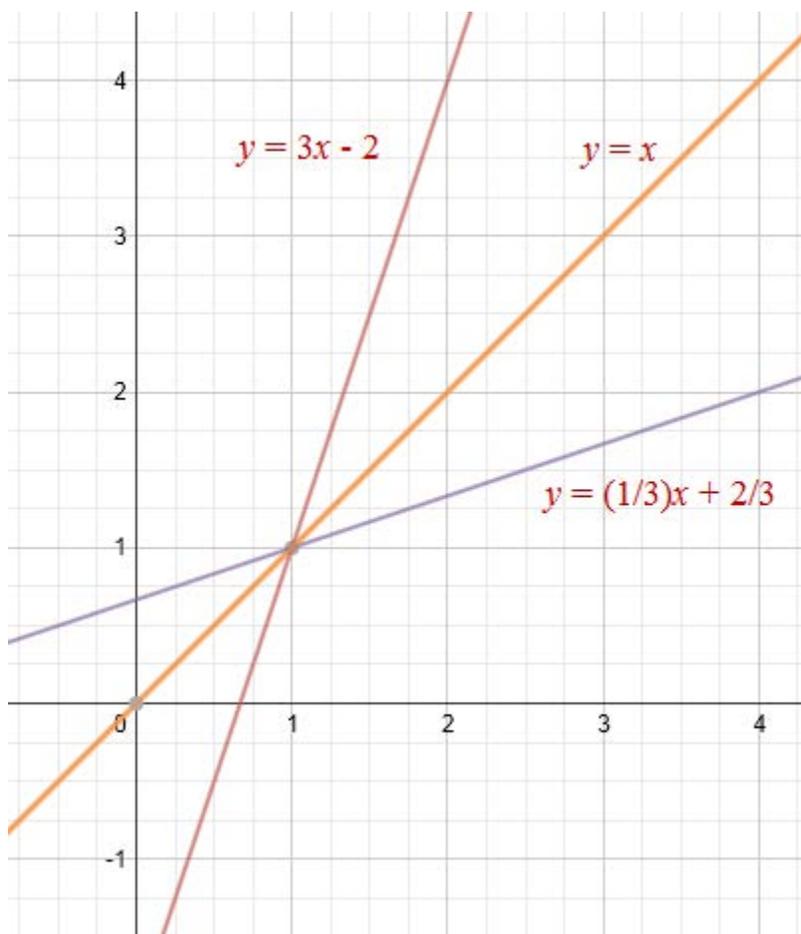
Divide each side by 3.

$$\frac{1}{3}x + \frac{2}{3} = y$$

Write $\frac{(1)x+2}{3}$ as two fractions.

The inverse of $y = 3x - 2$ is $y = \frac{1}{3}x + \frac{2}{3}$.

Now, let's graph the equations. Notice that the graphs are inverses of each other and reflected across the line, $y = x$.



Sometimes the equation is written in function notation as in the following two examples. To keep the computations simple, we remember that $f(x)$ can be written as y and vice versa.

Example #2: Find the inverse $f(x) = \frac{x+8}{4}$

$$f(x) = \frac{x+8}{4}$$

$$y = \frac{x+8}{4}$$

$f(x)$ can be written as y

$$x = \frac{y+8}{4}$$

Switch the x and y .

$$4x = y+8$$

Multiply each side by 4 $\left(\frac{y+8}{\cancel{4}_1} \cdot \frac{\cancel{4}^1}{1} = \frac{y+8}{1} = y+8 \right)$

$$4x - 8 = y$$

Subtract 8 from each side.

The inverse function of $f(x) = \frac{x+8}{4}$ is $y = 4x - 8$.

*The inverse of a function f is denoted by f^{-1} . This is read as “ f inverse” or “the inverse of f ”.

Thus, we can state that $f^{-1}(x) = 4x - 8$ is the inverse of $f(x) = \frac{x+8}{4}$.

Example #3: Find the inverse of $f(x) = \frac{5}{x-2}$.

$$f(x) = \frac{5}{x-2}$$

$$y = \frac{5}{x-2}$$

Write $f(x)$ as y .

$$x = \frac{5}{y-2}$$

Switch the x and y .

$$x(y-2) = 5$$

Multiply each side by $(y-2)$.

$$\text{Right side: } \left(\frac{5}{\cancel{x-2}_1} \cdot \frac{\cancel{x-2}^1}{1} = \frac{5}{1} = 5 \right)$$

$$xy - 2x = 5$$

Distributive Property

$$xy = 5 + 2x$$

Add $2x$ to each side.

$$y = \frac{5+2x}{x}$$

Divide each side by x .

$$\text{Left side: } \left(\frac{\cancel{x}^1 y}{\cancel{x}^1} = \frac{y}{1} = y \right)$$

$$y = \frac{5}{x} + \frac{2x}{x}$$

Write $\frac{5+2x}{x}$ as two fractions.

$$y = \frac{5}{x} + 2$$

$$\text{Simplify. } \left(\frac{2\cancel{x}^1}{\cancel{x}^1} = \frac{2}{1} = 2 \right)$$

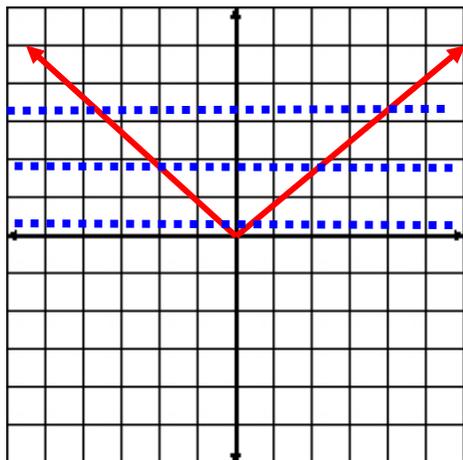
$$f^{-1}(x) = \frac{5}{x} + 2$$

Write the function using inverse notation.

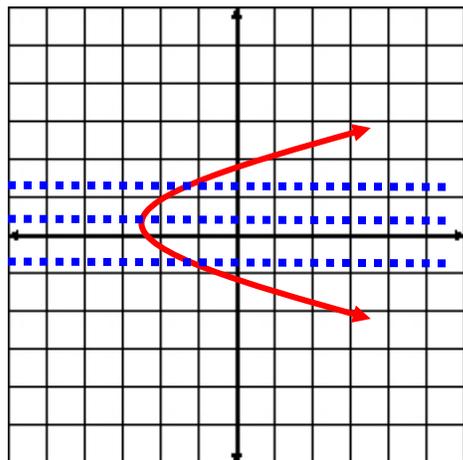
The inverse of $f(x) = \frac{5}{x-2}$ is $f^{-1}(x) = \frac{5}{x} + 2$.

The **horizontal line** test is a quick test used to determine if the inverse of a function is a function graphically.

The inverse of a function is a function, *if and only if*, every horizontal line intersects the graph of the given function at no more than one point.

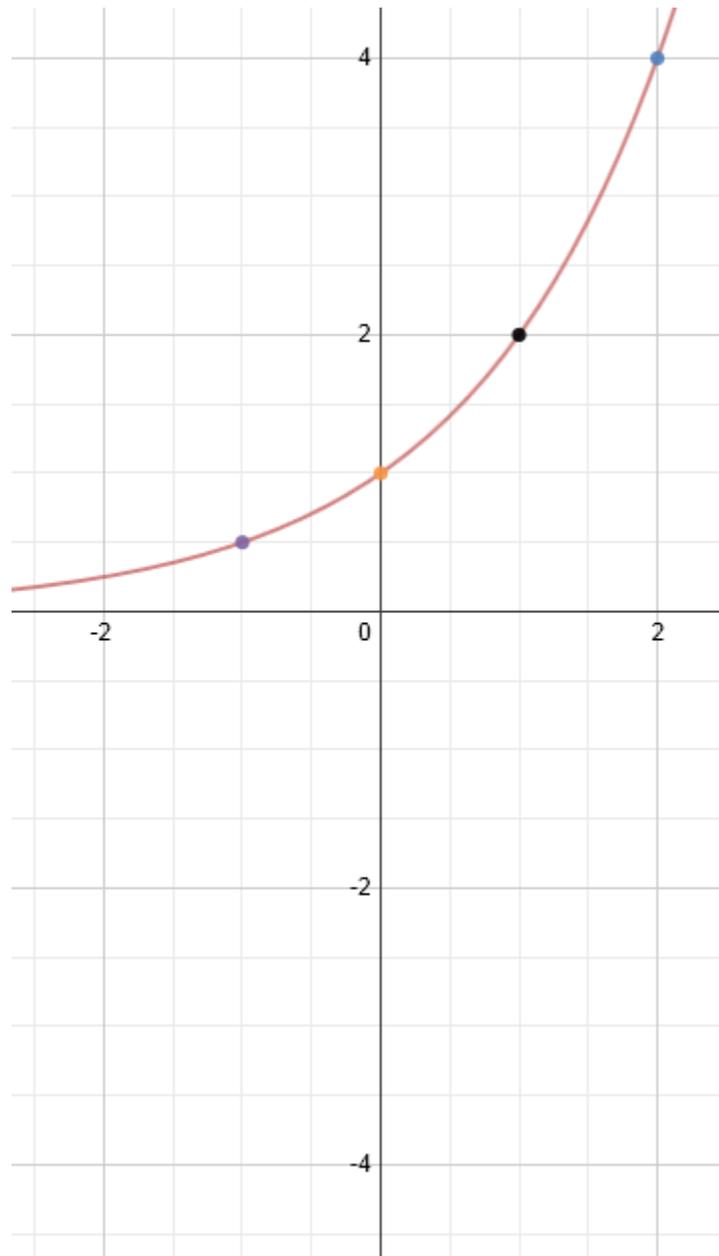


The horizontal lines intersect the graph at **more than 1** point. This means the inverse of the function is **not** a function.



The horizontal lines intersect the graph at only **one** point. This means the inverse of the relation is a function.

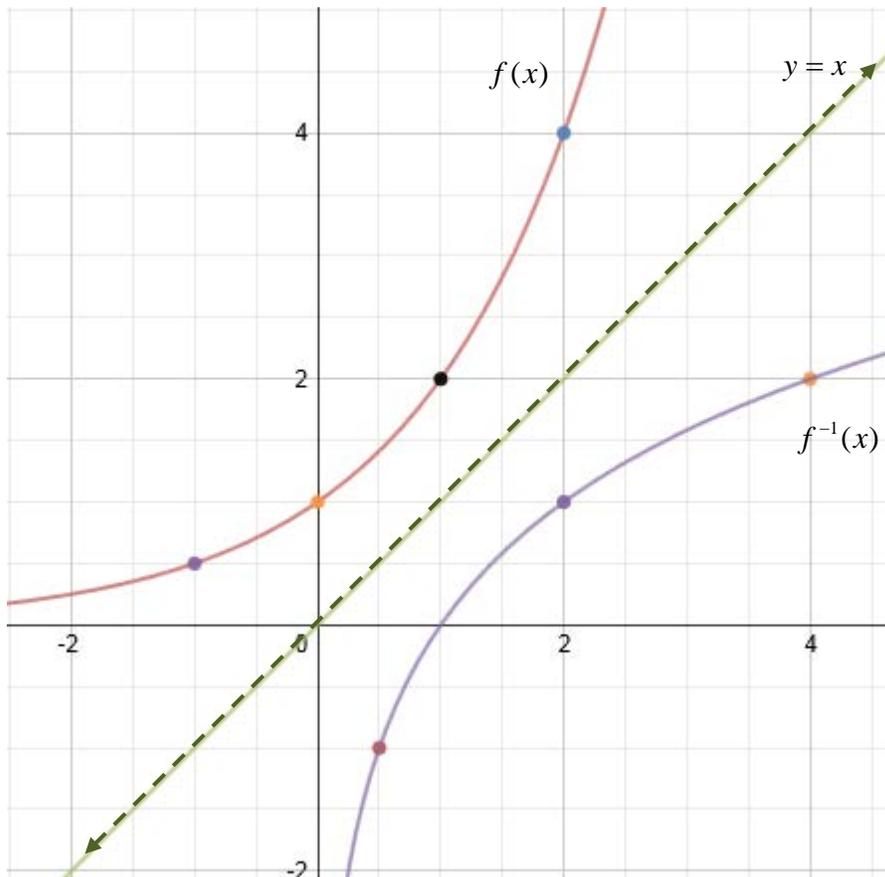
Example #4: Given the graph of function f as shown, sketch the graph of f^{-1} . Is the inverse relation a function?



Solution: Because the graph of f passes through the points $(-1, 0.5)$, $(0, 1)$, $(1, 2)$, and $(2, 4)$, the graph of f^{-1} must pass through the points $(0.5, -1)$, $(1, 0)$, $(2, 1)$ and $(4, 2)$. *Plot the points and draw a smooth curve through the points.

*We switch the x and y values for the inverse.

The graph looks like this.



Notice that the graph of f is symmetric to the graph of f^{-1} with respect to the graph of the line $y = x$. If the graph were folded along the dashed line, the graph of f would lie on top of the graph of f^{-1} . This is a characteristic of all graphs of functions and their inverses.

Test the reflection characteristic of functions and their inverses. Click [here](#) to view and print out the graph. Fold the paper along the dotted line, $y = x$.

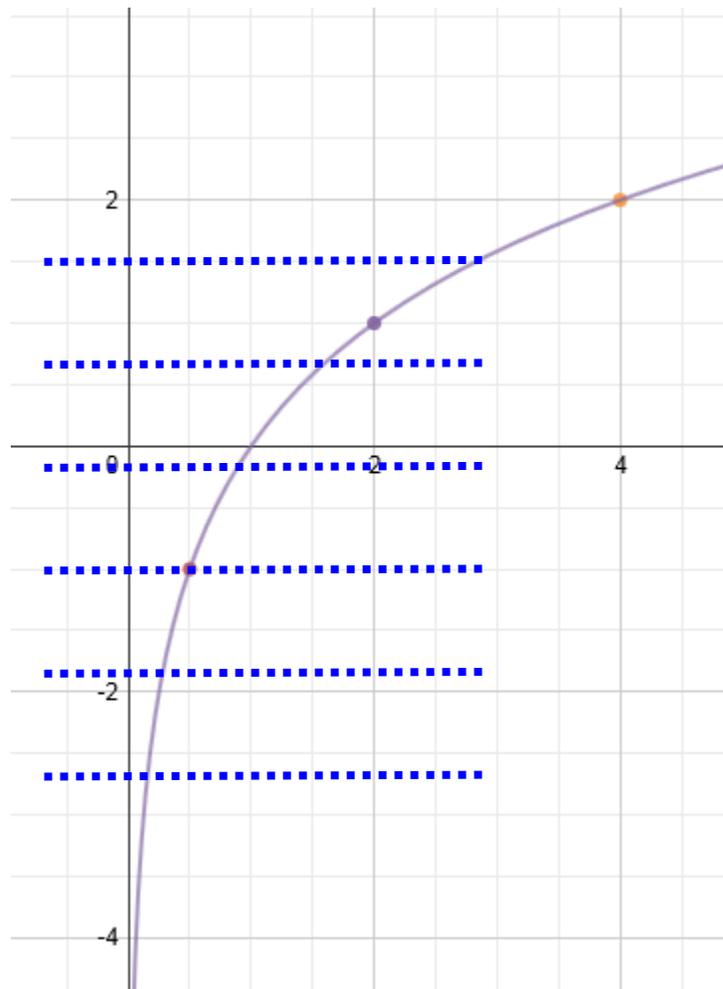


Does the graph of f lie on top of the graph of f^{-1} ?

“Click here” to check your answer.

Yes!

To determine whether f^{-1} is a function, use the horizontal line test.



Any horizontal line drawn through the graph of f^{-1} will intersect the graph at only one point. Thus, f^{-1} IS a function.

Example #5: Graph the following functions on one set of coordinate axes.

$$y = 2x + 4 \qquad y = \frac{1}{2}x - 2 \qquad y = x$$

To practice graphing functions and their inverses go to www.desmos.com/calculator or use a graphing calculator.



Are the functions reflections of each other over $y = x$?

“Click here” to check your answer.

Yes!

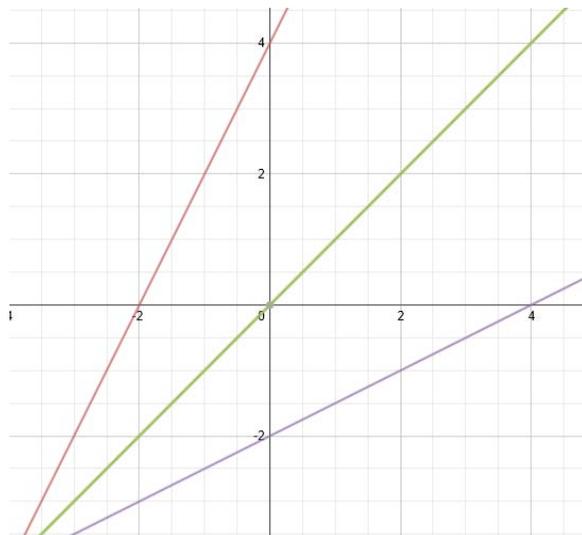


Are $y = 2x + 4$ and $y = \frac{1}{2}x - 2$ inverses of each other?

“Click here” to check your answer.

Yes!

Notice that the graphs of the functions are symmetric to the line $y = x$ and inverses of each other.



Another way to determine whether two functions are inverses of each other is by finding both of their compositions.

If f and g are inverse functions, then $f(g(x))$ and $g(f(x))$ will both $= x$.

Example #6: Determine if the given functions are inverses of each other by finding their compositions.

$$f(x) = 4x - 3$$

$$g(x) = \frac{1}{4}x + \frac{3}{4}$$

Substitute $g(x)$ in for x .

Substitute $f(x)$ in for x .

$$f(g(x)) = 4\left(\frac{1}{4}x + \frac{3}{4}\right) - 3$$

$$g(f(x)) = \frac{1}{4}(4x - 3) + \frac{3}{4}$$

Distribute

$$f(g(x)) = x + 3 - 3$$

$$g(f(x)) = x - \frac{3}{4} + \frac{3}{4}$$

Simplify

$$f(g(x)) = x$$

$$g(f(x)) = x$$

Since both compositions equal x , they are inverses of each other.

Example #7: Determine if the given functions are inverses of each other by finding their compositions.

$$f(x) = 3x$$

$$g(x) = \frac{1}{3x}$$

Substitute $g(x)$ in for x .

Substitute $f(x)$ in for x .

$$f(g(x)) = 3\left(\frac{1}{3x}\right)$$

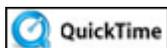
$$g(f(x)) = \frac{1}{3(3x)}$$

Multiply

$$f(g(x)) = \frac{3}{3x} = \frac{1}{x}$$

$$g(f(x)) = \frac{1}{9x}$$

Since $f(g(x)) \neq x$, the two functions are not inverses of each other. We did not test $g(f(x))$ because if either composition does not equal x , then we know that the functions are not inverses of each other.



Inverse Functions (05:36)

Stop! Go to Questions #15-33 to complete this unit.