MORE QUADRATIC FUNCTIONS

This unit is a review of previous units about quadratic functions.

Introduction to Quadratic Functions

Solving Quadratic Equations

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Introduction to Quadratic Functions

Quadratic functions have the form $f(x) = ax^2 + bx + c$ where the highest exponent is 2.

ax^2 is the quadratic term

bx is the linear term

c is the constant term

parabola: the graph of a quadratic function

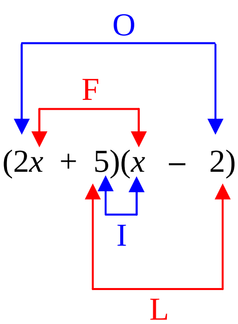
axis of symmetry: a line that divides the parabola into two parts that are mirror images of each other.

vertex: either the lowest point on the graph or the highest point on the graph.

domain of any quadratic function: the set of all real numbers

range: all real numbers ≥ the minimum value of the function (when opening up) or all real numbers ≤ the maximum value of the function (when opening down).

If given a function, such as $f(x) = (2x + 5)(x - 2)$, and asked to express it into quadratic form, use FOIL (First Outer Inner Last) multiplication to write it in the form $ax^2 + bx + c$.



Example #1: $f(x) = (2x + 5)(x - 2)$

$f(x) = 2x^2 + x - 10$

$a = 2, b = 1, c = -10$

By examining "*a*" in $f(x) = ax^2 + bx + c$, you can identify whether the function has a maximum value (opens up) or a minimum value (opens down).

If a > 0, the graph opens up and the *y*-coordinate of the vertex is the minimum value of the function *f*.

If a < 0, the graph opens down and the *y*-coordinate of the vertex is the maximum value of the function *f*.

Example #2: Determine if each function has a maximum value (opens down) or a minimum value (opens up).

a.) $f(x) = -3a^2 + 3a - 7$

-the value of "*a*" in this function is –3, so this function has a **maximum** value and opens **down**.

b.) $f(x) = (2x+1)(x-3)$

-first multiply the binomials to get $f(x) = 2x^2 - 5x - 3$, the value of "a" is 2, so this function has a **minimum** and opens **up**.

Solving Quadratic Equations

To solve quadratic equations
isolate the quadratic term
find the square root of each side

Example #1: $2x^2 + 14 = 50$

$$2x^2 = 36$$

$$x^2 = 18$$

$$x = \pm\sqrt{18}$$

Example #2: Solve for *x*: $9(x-2)^2 = 121$

$$\frac{9(x-2)^2}{9} = \frac{121}{9}$$

$$(x-2)^2 = \frac{121}{9}$$

$$\sqrt{(x-2)^2} = \sqrt{\frac{121}{9}}$$

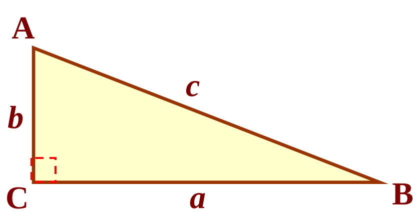
$$(x-2) = \pm\frac{11}{3}$$

$$x = \frac{11}{3} + 2 \qquad x = -\frac{11}{3} + 2$$

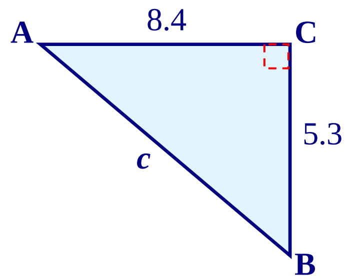
$$x = \frac{17}{3} \qquad x = -\frac{5}{3}$$

The Pythagorean Theorem

If $\triangle ABC$ is a right triangle with a right angle at C, then $a^2 + b^2 = c^2$.



Example #3:



$$(5.3)^2 + (8.4)^2 = c^2$$

$$28.09 + 70.56 = c^2$$

$$98.65 = c^2$$

$$\sqrt{98.65} = \sqrt{c^2}$$

$$9.9 \approx c$$

Factoring Quadratic Expressions

Summary of Techniques for Factoring Quadratic Expressions

Two terms

-look for a greatest common factor

Example #1: $4x^3 + 20x^2$ GCF is $4x^2$, factor this out of both terms.

$4x^2(x+5)$

Three terms (leading coefficient = 1 or –1)

-look for a greatest common factor, then

-find two factors of the constant term that when added together result in the middle term.

*If the last term is **positive**, then **both** factors will have the **same sign**, and that sign will be the sign of the middle term.

*If the last term is **negative**, then **one factor is positive** and the **other is negative**.

*If the leading coefficient is negative, factor out a –1 first, and then proceed to find factors of the last term that add up to the middle term.

Example #2: $x^2 - 9x - 22$ factors of –22 that add up to –9 are –11 and +2

$(x - 11)\,(x + 2)$

Difference of Squares (two terms)

-look for a GCF

-the first term will be a perfect square

-the last term will be a perfect square

-the terms will be separated with a subtraction sign

(square root of the first term + square root of the last term) (square root of the first term – square root of the last term)

Example #3: Factor: $4x^2 - 25$

$$(2x + 5)(2x - 5)$$

Three terms (leading coefficient > 1) *Trial and Error*

-look for a GCF

-factor the first term

-factor the second term

-the sum of the outside product and inside product must equal the middle term

Example #4: Factor $6x^2 - x - 2$

factors of $6x^2$	factors of -2
x and $6x$	1 and -2
$2x$ and $3x$	-1 and 2

Combine the factors so that the sum of the product of the outside terms and the product of the inside terms will = -1, the middle term.

$(2x + 1)(3x - 2)$

Product of outside terms = $-4x$

Check: $(2x + 1)(3x - 2)$

Product of the inside terms = $+3x$

The sum of the product of the outside terms and the product of the inside terms is

$$-4x + 3x = -x$$

which is the middle term

Zero Product Property

Zero Product Property

If $xy = 0$, then $x = 0$ or $y = 0$.

This property is used to find zeros of a function.

A zero of a function f is any number r such that $f(r) = 0$, or the solution.

To use the zero product property

1.) set the quadratic equal to zero

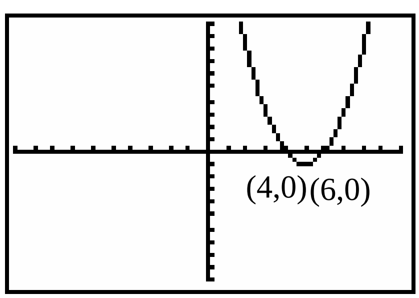
2.) factor the quadratic

3.) set each factor equal to zero and solve

Example #5:	$x^2 - 10x = -24$	add 24 to both sides of the equation
	$x^2 - 10x + 24 = 0$	factor the trinomial
	$(x - 4)(x - 6) = 0$	set each of these factors = to 0
	$x - 4 = 0$ and $x - 6 = 0$	solve each of these for x
	$x = 4$ and $x = 6$	

The zeros of this function are 4 and 6. These values mean that the function crosses the x-axis at $x = 4$ and $x = 6$.

The graph of this function is illustrated below and verifies the solution.



Completing the Square

When a quadratic equation does not contain a perfect square, you can create a perfect square in the equation by *completing the square*. **Completing the square** is a process by which you can force a quadratic expression to factor.

1.) make sure the quadratic term and the linear term are the only terms on one side of the equation (move the constant term to the other side)

2.) the coefficient of the quadratic term must be one,

3.) take one-half of the linear term and square it

4.) add this number to both sides of the equation

5.) factor the perfect square trinomial

6.) solve the equation

Example #1: Complete the quadratic expression into a perfect square.

$$x^2 - 20x$$

$$x^2 - 20x + 100 \qquad \frac{1}{2}(20) = 10, \quad 10^2 = 100$$

$$(x-10)^2$$

The completed perfect square is $x^2 - 20x + 100$ *or* $(x-10)^2$.

Example #2: Solve for x by completing the square.

$$x^2 + 6x - 16 = 0$$
$$x^2 + 6x = 16$$
$$x^2 + 6x \underline{+9} = 16 \underline{+9}$$

$$\frac{1}{2}(6) = 3 \rightarrow 3^2 = 9$$

$$(x+3)^2 = 25$$
$$\sqrt{(x+3)^2} = \sqrt{25}$$
$$x + 3 = \pm 5$$

$$x = 5 - 3 \quad \text{and} \quad x = -5 - 3$$
$$x = 2 \quad \text{and} \quad x = -8$$

Example #3: Solve for x by completing the square.

$$x^2 - 10x + 21 = 0$$
$$x^2 - 10x = -21$$
$$x^2 - 10x + \underline{\quad\quad} = -21 + \underline{\quad\quad}$$

fill in the blank with $\frac{1}{2}$ of 10, squared

$$x^2 - 10x + \underline{25} = -21 + \underline{25}$$
$$(x-5)^2 = 4$$
$$\sqrt{(x-5)^2} = \sqrt{4}$$
$$x - 5 = \pm 2$$

$$x = 2 + 5 \quad \text{and} \quad x = -2 + 5$$
$$x = 7 \quad \text{and} \quad x = 3$$

*If the coefficient of the quadratic term is not 1, you must **divide all terms** by the coefficient to make it one.

Example #4: Solve for *x* by completing the square.

$3x^2 - 6x = 5$ divide all terms by 3

$$\frac{3x^2}{3} - \frac{6x}{3} = \frac{5}{3}$$

$$x^2 - 2x \underline{+1} = \frac{5}{3} \underline{+1}$$

$$(x-1)^2 = \frac{8}{3}$$

$$\sqrt{(x-1)^2} = \sqrt{\frac{8}{3}}$$

$$x - 1 = \pm\sqrt{\frac{8}{3}}$$

$$x = \sqrt{\frac{8}{3}} + 1 \quad \text{and} \quad x = -\sqrt{\frac{8}{3}} + 1$$

$$x \approx 2.63 \quad \text{and} \quad x \approx -.63$$

Vertex Form of a Quadratic Function

$y = a(x-h)^2 + k$ where the vertex is located at (h, k) and the axis of symmetry is $x = h$

To write a quadratic in vertex form, complete the square first, using quadratic and linear terms only, if the coefficient of the quadratic term is 1.

Example #1: $y = -x^2 + 6x + 3$

$$y - 3 + \_\_\_ = -1(x^2 - 6x + \_\_\_)$$

$$y - 3 + (-9) = -1(x^2 - 6x + 9)$$

the –9 on the left came from multiplying the factored out –1 and the 9 from completing the square

$$y - 12 = -1(x-3)^2$$

$$y = -(x-3)^2 + 12$$

*If the leading coefficient is not one, factor the coefficient out of the quadratic and linear terms only.

Example #2: $y = -3x^2 - 6x - 7$

$$y + 7 + \_\_\_ = -3(x^2 + 2x + \_\_\_)$$

$$y + 7 + (-3) = -3(x^2 + 2x + 1)$$

the –3 on the left came from multiplying the factored out –3 and the 1 from completing the square

$$y + 4 = -3(x+1)^2$$

$$y = -3(x+1)^2 - 4$$

The Quadratic Formula

Summary of Quadratic Formula Techniques

The quadratic formula is used to solve any quadratic equation in standard form, $ax^2 + bx + c = 0$. The quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the quadratic formula

1.) make sure the equation is in standard form
2.) label the values of *a*, *b*, and *c*
3.) replace the values into the equation and solve

Summary for Finding the Axis of Symmetry and the Vertex of a Quadratic Function

*If a quadratic function is in standard form $ax^2 + bx + c = y$, then it is possible to locate the axis of symmetry by using the following

$$x = \frac{-b}{2a}$$

The axis of symmetry also refers to the *x*-value of the vertex. To find the *y*-value of the vertex:

1.) replace the value of *x* into the equation
2.) solve for *y*

See examples below of using the quadratic formula and finding the axis of symmetry and vertex for a quadratic function.

Example #1: Use the quadratic formula to solve the given quadratic for "*x*".

$x^2 - 16x - 36 = 0$ $\qquad a = 1,\ b = -16,\ c = -36$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(-36)}}{2(1)}$$

$$x = \frac{16 \pm \sqrt{256 + 144}}{2}$$

$$x = \frac{16 \pm \sqrt{400}}{2}$$

$$x = \frac{16 \pm 20}{2}$$

$$x = \frac{16 + 20}{2} \quad \text{and} \quad x = \frac{16 - 20}{2}$$

$$x = \frac{36}{2} \quad \text{and} \quad x = \frac{-4}{2}$$

$$x = 18 \quad \text{and} \quad x = -2$$

Example #2: Use the quadratic formula to solve the given quadratic for “*x*”.

$x^2 + 4x - 18 = 0$ $a = 1$, $b = 4$, $c = -18$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-18)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 + 72}}{2}$$

$$x = \frac{-4 \pm \sqrt{88}}{2}$$

$$x = \frac{-4 + \sqrt{88}}{2},\ x = \frac{-4 - \sqrt{88}}{2}$$

*These expressions can be simplified and this will be addressed in a later unit.

*If a quadratic function is in standard form, $ax^2 + bx + c = y$, then it is possible to locate the axis of symmetry by using the following formula:

Axis of Symmetry: $x = \frac{-b}{2a}$

Example #3: Find the axis of symmetry for the given quadratic.

$f(x) = 2x^2 + 8x + 19$ $a = 2$, $b = 8$, $c = 19$

The axis of symmetry is $x = \frac{-8}{2(2)} \rightarrow x = -2$

The axis of symmetry also refers to the *x*-value of the vertex.

To find the *y*-value of the vertex:

3.) replace the value of *x* into the equation
4.) solve for *y*

Example #4: Find the vertex of the parabola of the given quadratic.

$y = 2x - 2 + x^2$ — $a = 1$, $b = 2$, $c = -2$

$y = x^2 + 2x - 2$ — -put in standard form

-find the axis of symmetry

$x = \frac{-2}{2(1)}$ — -the axis of symmetry is $x = -1$

-replace all x values with –1 and solve for y

$$y = (-1)^2 + 2(-1) - 2$$
$$y = 1 - 2 - 2$$
$$y = -3$$

Therefore the vertex of this parabola will be located at (–1, –3).

The Discriminant

In the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression $b^2 - 4ac$ is known as the discriminant and will identify how many and what type of solutions there are to a quadratic equation.

Types of Solutions

If the value of $b^2 - 4ac$ is **positive**	2 real solutions
If the value of $b^2 - 4ac$ is **negative**	2 imaginary solutions
If the value of $b^2 - 4ac$ is **zero**	1 real solution

Example #1: Find the discriminant and determine the number of solutions for each of the quadratics shown below.

1.) $3x^2 - 6x + 4 = 0$
$b^2 - 4ac$
$(-6)^2 - 4(3)(4)$
36 – 48 = –12
2 imaginary solutions

2.) $4x^2 - 20x + 25 = 0$
$b^2 - 4ac$
$(-20)^2 - 4(4)(25)$
400 – 400 = 0
1 real solution

3.) $9x^2 + 12x = -2$
$b^2 - 4ac$
$(12)^2 - 4(9)(2)$
144 – 72 = 72
2 real solutions

*Note: In the third quadratic equation, express the quadratic equation in standard form, $9x^2 + 12x + 2 = 0$, to determine a = 9, b = 12, and c = 2.

Computing with Complex Numbers

To add or subtract complex numbers
- -combine the real parts
- -combine the imaginary parts

Example #1: Find the sum: $(-10-6i)+(8-i)$

$(-10+8)+(-6i-i)$

$-2-7i$

Example #2: Find the difference: $(-9+2i)-(3-4i)$

$(-9-3)+(2i-(-4i))$

$-12+6i$

To multiply complex numbers
- -use FOIL multiplication
- -combine like terms
- -change i^2 to (–1)

Example #3: Find the product: $(2-i)(-3-4i)$

$-6-8i+3i+4i^2$

$-6-5i+4(-1)$

$-6-5i-4$

$-10-5i$